Rieman Curvature

Recall Gaussian curvature of 2-plane is the product of 2 principal curvatures.

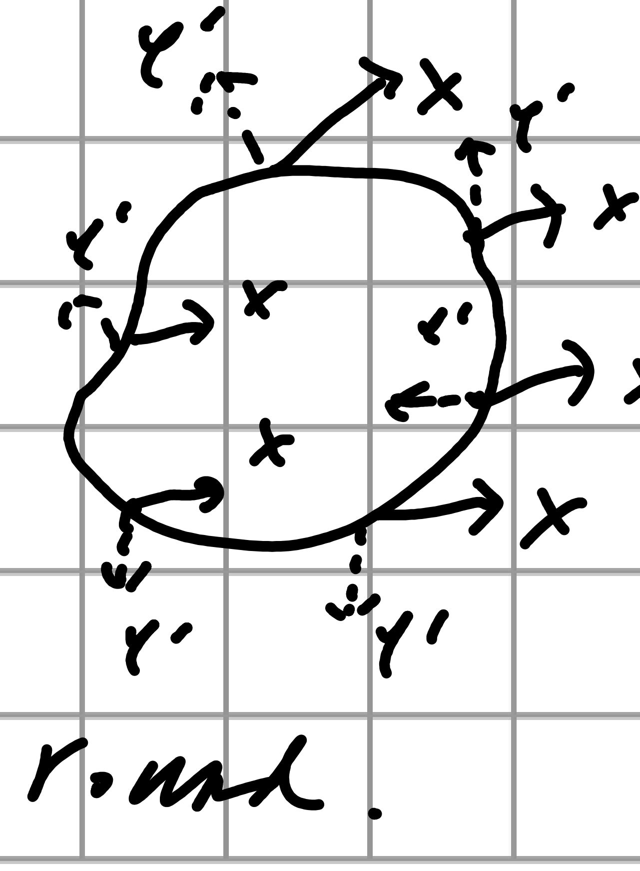
Riemann want to describe curv. of R-M (M^m, g) for general $m \geq 3$. by Gaussian curv.

<u>Rmk:</u> i) Let $U \subset T_pM$. s.t. $\exp_p : U \to V \subset M$ is diffeo. $N := \exp_p(U \cap \Pi)^{(*)}$ for 2-plane $\Pi \subset T_pM$. (i.e. union of geod.) Then Riemann define the sectional curvature $k(\Pi)$ as Gaussian curvature at p of N. (depend only on g)

(*). It's set of geodesic. $\Rightarrow$ flat test.

ii) Recall Gauss-Bonnet Thm:

$\int_D k \, dA = 2\pi - T(\partial D)$. measuring the integral of Gauss curvature $k(p)$. Note that RHS is rotation angle of holomy $P_{\partial D}$. i.e. compare para. transport X and tangent vec. γ' around ∂D. 2π is the rotation angle of γ' for 1 round.



while $T_C(\partial D)$ is rotation angle of X. (Since Y' return its initial value but X doesn't. !)

$\Rightarrow$ It can be used to find $k(p)$ by measuring holomy of small loops around p. (And divide by the area.)

i.e. holomy around ε-loop at p is a ε-rotation at speed $k(p)$. as $\varepsilon \to 0$.

(1) Definition:

Recall: $0 = X(Yf) - Y(Xf) - [X,Y]f$.

Note that $[X,Y]$ acts as a corrector when measuring symmetry of X, Y here.

We also want to measure sym of $(X,Y) \mapsto \nabla_X \nabla_Y Z$, fix LC connection ∇:

Def: i) Riemannian curv. op.: $R: \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ is $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Rmk: i) It's linear and tensorial (i.e.

$R(fX, Y)Z = R(X, fY)Z = R(X,Y)(fZ) = fR(X,Y)Z)$ by product rule of $[\,,\,]$ & ∇.. Also, its value at p only depend on X_p, Y_p, Z_p.

ii) We can generalize it on vector bundle $E \to M$: $R^{\nabla}(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \cdots$

<u>e.g.</u> On trivial bundle $M \times \mathbb{R}^1$:

$$R^{\nabla}(X,Y)\sigma = X(Y\sigma) - Y(X\sigma) - [X,Y]\sigma$$
$$= 0.$$

ii) Rieman curvature tensor $R(X, Y, Z, W) := \langle R(X,Y)Z, W\rangle$

<u>Rmk</u>: Under local chart (U, φ) and the coordinate frame $\{\partial_i\}$.

Set $R(\partial_k, \partial_i)\partial_l := \sum_m R_{lik}^{m} \partial_m$.

$$\langle R(\partial_k, \partial_i)\partial_l, \partial_j\rangle := R_{ljki}$$
$$= \sum_m R_{lik}^{m} g_{mj}$$

Write in Christoph symbols : We have

$$R_{ikl}^{j} = \partial_k \Gamma_{il}^{j} - \partial_l \Gamma_{ik}^{j} + \sum_m (\Gamma_{il}^{m} \Gamma_{mk}^{j} - \Gamma_{ik}^{m} \Gamma_{ml}^{j})$$

(2) Symmetry :

Thm. i) $R(X,Y) = -R(Y,X)$

ii) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$

iii) $\sum_{cyc} R(X,Y)Z = 0$. (1st Bianchi id.)

iv) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

Rmk: ii) means $R(X,Y)$ is truly a infi-nitesimal rotation.

Pf: i) is trivial. iv) follows from i) - iii) by cyclically permute $XYZW$.

For iii) : By linear and tensorial :

Set $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$

WLOG. $i \neq j \neq k$. otherwise, by i), it's trivial case.

So: LHS $= \sum \nabla_i (\underbrace{\nabla_j \partial_k - \nabla_k \partial_j}_{= [\partial_j, \partial_k]}) = 0$

For ii) : Set $X = \partial_i \neq \partial_j = Y$

We show $\langle R(X,Y)T, T\rangle = 0$.

(Then let $T = Z + W$ to obtain it)

It's from metric compat and torsion-free of ∇, $[\partial_j, \partial_i] = 0$

check $\partial_j(\partial_i \langle T, T\rangle) = \square$.

Rmk: We have sym bilinear form on $\Lambda_2 T_p M$

: $S(X\wedge Y, Z\wedge W) \overset{\Delta}{=} -\langle R(X,Y)Z, W\rangle$.

Def: Sectional curvature of 2-plane $\Pi \subset T_p M$

is $K(\Pi) := S(X\wedge Y, X\wedge Y) / \langle\langle X\wedge Y\rangle\rangle$

where $\langle\langle X\wedge Y, Z\wedge W\rangle\rangle := \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle$ and $\Pi = \mathrm{span}\{X, Y\}$.

Rmk: It agrees with Rieman's original idea $K(\Pi)$ is the Gaussian curv. of flattest surf. $N \subset M$ s.t. $T_pN = \Pi$.

Lemma. If S is sym. bilinear form on $\Lambda_2 V$ satisfies 1st Bianchi id. i.e.

$\sum_{cyc.}^{X,Y,Z} S(X\wedge Y, Z\wedge W) = 0$. Then:

$S(X\wedge Y, X\wedge Y) = 0 \ \forall X, Y \in V \Rightarrow S \equiv 0$.

Pf: 1) $0 = S((X+Y)\wedge Z)$

$= 2S(X\wedge Z, Y\wedge Z)$

2) $0 = S(X\wedge(Z+W), Y\wedge(Z+W))$

$= S(X\wedge Z, Y\wedge W) + S(X\wedge W, Y\wedge Z)$

$S_0 = S(X\wedge Z, Y\wedge W) \overset{\text{permute } XYZW}{=}$

$S(Z\wedge X, Y\wedge W) =$

$S(X\wedge W, Y\wedge Z)$

With 1^{st} Bianchi's id.

Rmk: S can actually be expressed in 16 terms of its diagonal:

$6S(X\wedge Y, Z\wedge W) = S((X+Z)\wedge(Y+W))\cdots$

Thm. The sectional curv. $K(\pi)$ of 2-plane $\pi \subset T_pM$ completely determines the Riemann curvature R_p at $p\in M$.

(3) 2^{nd} - Bianchi. id.:

Note connection can be defined on v.b.:

Def: $(\nabla_X R)(Y,Z)W := \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W$

$- R(Y, \nabla_X Z)W - R(Y,Z)(\nabla_X W)$.

Rmk: R's C^∞-linear on $X\,Y\,Z\,W$.

prop. (2^{nd} - Bianchi id.)

$$\sum_{cyc}^{XYZ} (\nabla_X R)(Y,Z)W = 0. \quad \forall X,Y,Z,W \in \mathfrak{X}(M)$$

Pf: WLOG. $\{X,Y,Z,W\} \subset \{\partial_i\}$.

Expand the three term with using $\nabla_X Y = \nabla_Y X$

Def: i) R-M (M^n, g) is isotropic at $p \in M$ if $K(\pi) = f(p)$. indept of choice of 2-plane $\pi \subset T_pM$.

ii) R-M (M^n, g) has const. curvature if $K(\pi) \equiv k$. indept of p and $\forall$ 2-plane $\pi \subset T_pM$.

Rmk: Set $R_1(X,Y)Z := \langle Y,Z\rangle X - \langle X,Z\rangle Y$

$\Rightarrow$ R_1 has const. curvature.

Since $-\langle R_1(X,Y)Z, W\rangle = \langle\langle X\wedge Y, Z\wedge W\rangle\rangle$

so $k \equiv 1$.

Lemma. $\nabla_X R_1 \equiv 0$. (by metric compt. of ∇)

Thm. (M^n, g) is connected R-M s.t. $n \geq 3$ which is isotropic at $\forall p \in M$. Then: M has const. Riemann curvature.

Rmk: It's false if $n=2$ since any surface is trivially isotropic.

Pf: We have $R(X,Y)Z = k_p R_1(X,Y)Z$.

$$\Rightarrow (\nabla_X R)(Y,Z)W \overset{\text{Lem}}{=} (Xk) R_1(Y,Z)W.$$

cyclically permute X, Y, Z.

using 2^{nd} Bianchi id. we got

$$\sum_{XYZ \text{ cyc}} \left((Zk)\langle Y,W\rangle - (Yk)\langle Z,W\rangle \right) X = 0. \quad (*)$$

Let X, Y, Z orthogonal. $\Rightarrow$ coeff $=0$

So: $0 = (Xk)\langle Z,W\rangle - (Zk)\langle X,W\rangle$

Set $Z = W \Rightarrow Xk = 0$. $\forall X \in \mathfrak{X}(M)$

So $k \equiv$ const on connected M.

Rmk: $(*)$ using $n \geq 3$!

Thm. Any two manifolds of same const. curvature k are locally isometric.

Rmk: Up to scaling, any mfd of nonzero curv. has curv. $k = \pm 1$.

e.g. $k_{S^m} = 1$. $k_{\mathbb{H}^m} = -1$. $k_{\mathbb{R}^m} = 0$

Thm.[(*)] Any complete connected mfd of const. curvature k is isometric to S^m or $\mathbb{R}^m$ or $\mathbb{H}^m$ (by some discrete. fixed-point free group of isometric) locally up to scaling.

(4) Examples for const. curv.

① Hyperbolic Space $\mathbb{H}^m$:

$\mathbb{H}^m$ is simply connected space with $k \equiv -1$. Next, we will def $\mathbb{H}^m$ as a open subset of $\mathbb{R}^m$

Set $\varphi : U \subset \mathbb{R}^m \to \mathbb{R}^1$ smooth. And give Rieman metric on U by $g_{ij} = e^{-2\varphi} \delta_{ij}$

Denote $\varphi_k = \partial_k \varphi$. Then for LC conn. ∇:

Yes $\Gamma_{ij}^k = -\varphi_i \delta_{jk} - \varphi_j \delta_{ik} + \varphi_k \delta_{ij}$

Also express R_{ikl}^{j} in (Γ_{ij}^k). take into

$R_{ijkl} = \sum_m g_{jm} R_{ikl}^{m} = e^{-2\varphi} R_{ikl}^{j}$

So: $K(\pi_{ij}) = R_{ijji} / g_{ii} g_{jj} = e^{-2\varphi} R_{iij}^{j}$

Now, let $U = \{x^1 > 0\}$ and $\varphi(x) = \log(x^1)$

$\Rightarrow K(\pi_{ij}) = -1$

And actually, in the formula above, we have:

$K = -K_1$. So: $K \equiv -1$.

<u>Rmk</u>: i) Geodesic in such $\mathbb{H}^m$ are rays &

semicircles $\perp \{x^1 = 0\}$

(Since we have Γ_{ij}^k. We can solve it)

ii) Equi. conformal model is set $U = B_1(0)$

and $\varphi(x) = \log(1-|x|^2) - \log 2$

iii) Isometry for $\mathbb{H}^m$ is mobius transf on

$\mathbb{R}^m$ preserve U.

② <u>Pinched Curvature</u>:

On last Thm(*) in (3). :

i) If we drop "completeness". Then M is too varied to describe.

ii) If we drop "simply conn". Then M is quotient of S^m by some discrete group of isometries. e.g. $\mathbb{R}P^m = S^m/\{\pm 1\}$.

Rmk: i) If $m = 2n$. then $\mathbb{R}P^m$ is the only

ii) If $m = 2n+1$, then there infinite many example, e.g. lens space $S^{2n+1}/\mathbb{Z}^k$. where $\mathbb{Z}^k = \{e^{2\pi i n/k}\}$ finite cyclic group. which has $K \equiv 1$.

What if we consider mfd with $K > 0$ and pinched between 2 const. ?

e.g. Complex proj. space $\mathbb{C}P^n \simeq S^{2n+1}/S^1$. with with nature metric has $1 \le K \le 4$.

Thm. If complete. simply connected mfd M^m has $0 < K(\pi) < 4K(\pi')$. $\forall p$. $\forall$ 2-planes π, $\pi' \in T_p(M)$. Then. M admits a metric with const. K. So it's diffeo to S^m.

(5) <u>Moving frame:</u>

Next, we consider general frame $\{E_i\}$ for T_pM, $p \in U \subseteq M$, rather than coordinate frame $\{\partial_i\}$

<u>Rmk</u>: i.e. $[E_i, E_j] \neq 0$ in general!

1) Find dual basis:

$\{\theta^i\} \subseteq T_p^*M$, s.t. $\theta^i(E_j) = \delta_{ij}$, $p \in U$.

Note $\nabla_{E_i} E_j = \sum \Gamma_{ij}^k E_k$.

So: $\Gamma_{ij}^k = \theta^k(\nabla_{E_i} E_j)$.

<u>Def</u>: Connection one-form $\theta_j^k := \sum_l \Gamma_{lj}^k \theta^l$

<u>Rmk</u>: Note $\theta_j^k(E_l) = \Gamma_{lj}^k$. By linear:

$\nabla_X E_j = \sum_k \theta_j^k(X) E_k$, i.e. ∇

Will be determined by $(\theta_j^k)_{m\times m}$.

$(E_1 \cdots E_m)^T \cdot (\theta_j^k) = (\nabla E_1 \cdots \nabla E_m)^T$

2) Express symmetry of ∇ in $\{\theta^k\}_k$

<u>Rmk</u>: Since $[E_i, E_j] \neq 0$. So symmetry of ∇ won't be equi. with $\Gamma_{ij}^k = \Gamma_{ji}^k$.

By formular for one-form $d\omega(X,Y)$:

$d\theta^k(E_i,E_j) = E_i\theta^k(E_j) - E_j\theta^k(E_i)$

$- \theta^k([E_i,E_j]) = -\theta^k([E_i,E_j])$

And $\theta^k(\nabla_j E_i - \nabla_i E_j) = \Gamma^k_{ji} - \Gamma^k_{ij}$

$= \sum_l \theta^l(E_i)\theta^k_l(E_j) - \theta^l(E_j)\theta^k_l(E_i)$

$= \sum_l \theta^l \wedge \theta^k_l(E_i,E_j)$.

So: if ∇ is torsion-free / symmetry.

$\Rightarrow d\theta^k = \sum_l \theta^l \wedge \theta^k_l$. $\forall k$.

3) Express metric compat. of ∇ in $\{\theta^k\}$:

Denote $g_{ij} = g(E_i, E_j)$. if g compatible:

$dg_{ij}(E_k) = E_k g_{ij}$

$= g(\nabla_{E_k}E_i, E_j) + g(E_i, \nabla_{E_k}E_j)$.

$= \sum_l (g_{jl}\theta^l_i(E_k) + g_{il}\theta^l_j(E_k))$

Set $\theta_{ij} := \sum_k g_{kj}\theta^k_i$. So we have:

$dg_{ij} = \theta_{ij} + \theta_{ji}$.

4) Express Rieman curvature in $[\theta_k]$:

Denote $R(E_k, E_l)E_i =: \sum_j R_{ikl}^j E_j$.

Def: i) curvature two form $\Omega_i^j := \sum_{k<l} R_{ikl}^j \theta^k \wedge \theta^l$

Rmk: $R(X,Y)E_i = \sum \Omega_i^j (X,Y) E_j$ by bilinearity.

So: $(E_1 \cdots E_m)^T \cdot (\Omega_i^j) = (R(\ ,\ )E_1 \cdots R(\ ,\ )E_m)^T$

ii) $\Omega_{ij} := \sum_k g_{kj} \Omega_i^k = \sum_{k<l} R_{ijkl} \theta^k \wedge \theta^l$

Rmk: $\Omega_{ij} = -\Omega_{ji}$.

Thm. $\Omega_i^j = d\theta_i^j + \sum_k \theta_k^j \wedge \theta_i^k$.

Rmk: i.e in matrix: $\Omega = d\theta + \theta \wedge \theta$.

Pf. Simply check $\sum_j \Omega_i^j (X,Y) E_i$

$= \sum_j (d\theta_i^j (X,Y) - \cdots) E_i$

with formula w.r.t. $dW(X,Y)$ = $\square$

Cor. For (e_i) o.n.b. $\subset TM$ (i.e. $g_{ij} = \delta_{ij}$)

with coframe (w_i). We have:

$$\Omega_i^j = dw_i^j - \sum w_i^k \wedge w_k^j$$

<u>Rmk</u>: By Gram-Schmidt method, we can construct such $\{e_i\}$.

Then: $g = \sum w^i \otimes w^i$ & 2-form.

As above, to charac. LC ∇:

$dw^i = \sum_j w^j \wedge w_j^i$, $w_j^i + w_i^j = 0$.

And $w_{ij} = \sum_k g_{kj} w_i^k = w_i^j$.

with $\Omega_{ij} = \sum g_{kj} \Omega_i^k = \Omega_i^j$.

<u>prop.</u> For M^2 with orthogonal frame $\{e_i\}$.

We have $dw_1^2 = \Omega_1^2 = -k w^1 \wedge w^2$. So

k is Gauss curvature.

<u>Pf</u>: $\Omega_1^2 = dw_1^2 - \sum_{1,2} w_1^k \wedge w_k^2 = dw_1^2$.

Since $w_i^j + w_j^i = 0 \Rightarrow w_1^1 = w_2^2 = 0$

And $\Omega_1^2 = \Omega_{12} = \sum_{k<l} R_{12kl} w^k \wedge w^l$

$= R_{1212} w^1 \wedge w^2$

$k = k(\pi) = \langle R(e_1, e_2) e_1, e_2 \rangle$

$= -R_{1212}$

Thm. (M,g) is R-m. $\Pi \subset T_pM$ is a 2-plane.
For ε small enough, so $N = \exp_p(\Pi \cap B_\varepsilon(0))$
has $T_pN = \Pi \subset T_pM$. Then: Sectional curva.
$K(\Pi)$ = Gauss curva. K of N at p.

Pf: 1) Set $(U_p, x_1 \cdots x_m)$ is normal coordinate.
$\{\partial_i\}$ is coordinate frame. $\Pi = \mathrm{span}\{\partial_1, \partial_2\}$
Apply Gram-Schmidt method on $\{\partial_i\}$
to get $\{e_i\}$ o.n.b. Note: $\begin{cases} e_1 = a_1^1 \partial_1 + a_1^2 \partial_2 \\ e_2 = a_2^1 \partial_1 + a_2^2 \partial_2 \end{cases}$
$a_i^j(p) = \delta_{ij}$ $(g_{ij}(p) = \delta_{ij})$.
$\nabla_{X_p} e_i = \sum \omega_i^j(X_p)\partial_j = (X_p a_i^1)\partial_1 + (X_p a_i^2)\partial_2$
for $i = 1, 2$. Since $\nabla_{X_p}\partial_i = 0$, $\forall i$ $(\Gamma_{ij}^k(p) = 0)$
So: $\omega_i^j(X_p) = 0$. for $i = 1, 2$. $j \geq 3$.

2) $\iota : N \to M$. inclusion. $\tilde{\omega}^i := \iota^* \omega^i$. $\tilde{\omega}_i^j = \iota^* \omega_i^j$
So: $\tilde{\omega}^i = 0$ for $i > 2$.
$$-\tilde{\omega}_j^i = \tilde{\omega}_i^j \qquad d\tilde{\omega}^i = \sum \tilde{\omega}^j \wedge \tilde{\omega}_j^i.$$
from ι^* commutes with $\wedge$, d.

3) By prop. above: $d\tilde{\omega}_1^2 = -K\tilde{\omega}^1 \wedge \tilde{\omega}^2$.
$$d\omega_1^2 = \sum \omega_1^k \wedge \omega_k^2 + \sum_{k<l} R_{12kl}\, \omega^k \wedge \omega^l$$
operate ι^* on both side: $d\tilde{\omega}_1^2 = R_{1212}\tilde{\omega}^1 \wedge \tilde{\omega}^2$.