Flat Metric

<u>Def</u>: R-M (M, g) is flat if its Rieman curvature $R \equiv 0$.

<u>Thm.</u> $f: M \to M$. diffeo. $X \in \mathfrak{X}(M)$ with local flow θ_t and integral curve γ_p

Then: i). ii). iii) equi. below,

i) $X_{f(p)} = f_* X_p$ ii) $f \circ \gamma_p = \gamma_{f(p)}$

iii) $f \circ \theta_t = \theta_t \circ f$.

<u>Pf</u>: ii) $\Leftrightarrow$ iii) is trivial. For i) $\Leftrightarrow$ ii):

$\gamma'(t) = X_{\gamma(t)} \Rightarrow f_* \gamma'(t) = f_* X_{\gamma(t)}$

i) $\Rightarrow$ ii): $X_{f \circ \gamma(t)} = f_* X_{\gamma(t)}$

$= f_* \gamma_* (\frac{d}{dt}) = (f \circ \gamma)_* (\frac{d}{dt})$

So: $f \circ \gamma(t)$ is flow pass $f(p)$. at $t = 0$.

By uniqueness. $\gamma_{f(p)}(t) = f \circ \gamma(t)$.

ii) $\Rightarrow$ i): reverse the argument.

Consider the infinitesimal of X.

Cor[1]: χ_s, θ_t are local flow of $X, Y \in \mathfrak{X}(M)$. Then: $[X, Y] = 0 \Leftrightarrow \forall p \in M$. $\chi_s \circ \theta_t(p) = \theta_t \circ \chi_s(p)$. for $|t|, |s|$ small.

Pf: Set $f(t) = \chi(t)$. above.

Using def of Lie deri. L_X and from $L_X Y = [X, Y]$.

(($\Leftarrow$) trivial. ($\Rightarrow$) Check $Z_p(t) = \theta_{-t*}(Y_{\theta_t(p)})$ has 0 derivative)

Cor[2]: $\{X_i\}$ is frame of $\mathfrak{X}(M)$. s.t. $[X_i, X_j] = 0$. Then $\forall p \in M$. $\exists$ local chart (U, φ). s.t. $X_i = \partial_i$ is the coordinate frame.

Rmk: It's like the compatible cond in PDE. e.g. $f_x = g$. $f_y = h$ can be solved needs $g_y = h_x$.

Pf: Denote $\theta_t^i(p)$ is flow of X_i.

First set $\varphi(\theta_t^1(p)) := (t, 0 \cdots 0)$

inductively, set

$\varphi^{-1}(t_1, \dots t_m) = \theta^m_{t_m}(\dots \theta^2_{t_2}(\theta^1_{t_1}(p))\dots)$

By commute of θ^i_t. Differentiate both side w.r.t. t_i. We can see that $X_i = \partial_i$

Def: i) k-plane distribution Δ on M is a smooth choice of k-dim subspace $\Delta_p \subset T_pM$ each tangent space

ii) Δ in i) is involutive if $X_p, Y_p \in \Delta_p$ $\forall p \in M \Rightarrow [X, Y]_p \in \Delta_p$, $\forall p \in M$.

Rmk: 2-plane dist. $\Delta = \mathrm{span}\{\partial_1, \partial_2 + x^1 \partial_3\}$ on $\mathbb{R}^3$ is nowhere involut.

(e.g. $x=0$: $X_i = \partial_i$, $i \le 2$. $[X_1, X_2] = \partial_3$)

which forms contract structure on $\mathbb{R}^3$.

iii) Δ in i) is completely integrable if $\forall p \in M$, $\exists$ k-dim submfd N. st. $p \in N$. $T_qN = \Delta_q$, $\forall q \in N$.

Thm. (Frobenius)

Completely integrable $\Leftrightarrow$ involutive.

Pf: ($\Rightarrow$) is trivial. For ($\Leftarrow$):

We show $\forall p \in M$. $\exists k$ commuting v.f. spans A_p. Then, we can construct coordinate charts for N by using flow as in Cor².

Thm. (M^n, g) is locally isometric to $\mathbb{R}^n$

$\Leftrightarrow$ M is flat.

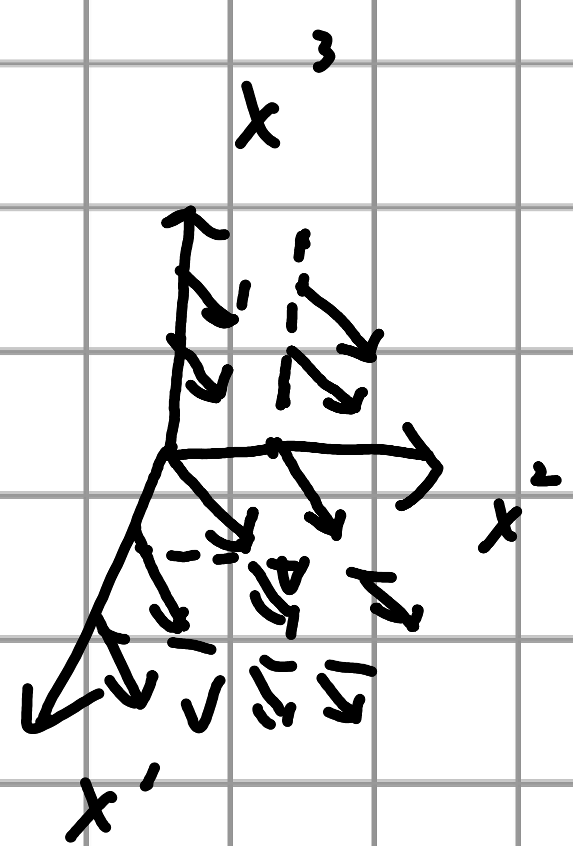
Lemma*. ∇ is connection on v.b.: $E \to M$.

Then: $R^\nabla \equiv 0 \Leftrightarrow \forall p \in M$. it locally have frame $\{\sigma_a\}$ s.t.

$\nabla_X \sigma_a = 0$. $\forall X \in \mathfrak{X}(M)$. $\forall a$.

Rmk: R^∇ can be seen as the compatible cond. for exist of parallel section $\{\sigma_a\}$.

Pf: ($\Leftarrow$) is trivial by linear & tensorial.



For $(\Rightarrow)$: We work locally around p. Set $\{\sigma_\alpha\}$ is a basis for E_p.
First, we parallel transport each σ_α along x^1-axis. Start from each point of line $p+x^1$. We parallel transp. each σ_α along x^2-axis. Keep doing m times. untill them def on M.

We inductively prove: $\nabla_{\partial_i}\sigma_\alpha = 0$. $1\le i\le k$ on $(x^1,\dots,x^k)$-plane M_k.

$k=1$ is trivial. For $n=k+1$.

$\nabla_{\partial_{k+1}}\sigma_\alpha = 0$ by def. With $R^\nabla(\partial_i,\partial_{k+1}) = 0$

So: $\nabla_{\partial_{k+1}}\nabla_{\partial_i}\sigma_\alpha = \nabla_{\partial_i}\nabla_{\partial_{k+1}}\sigma_\alpha = 0$

By induction hypo. $\nabla_{\partial_i}\sigma_\alpha = 0$ on M_k

$\Longrightarrow \nabla_{\partial_i}\sigma_\alpha = 0$ on M_{k+1}. by uniqueness

So: $\nabla_{\partial_i}\sigma_\alpha = 0$. $\forall\, i\le m$. $\forall \alpha$.

<u>Pf</u>: $(\Rightarrow)$ LC connection on $\mathbb{R}^m$ is $\nabla_X Y = \sum_{ij} X^i \partial_i (Y^j)\,\partial_j$ $(\Gamma_{ij}^k = 0)$.

So $R^{\nabla^{\mathbb{R}^m}} = 0$.

Then pull back $\nabla^{\mathbb{R}^m}$ on M by isometry

is still a LC connection.

(E) Start from o.n.b. of T_pM and

apply the Lem.* to extend them to

parallel frame $\{E_i\}$ on nbd U_p.

So: $[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i = 0$.

Then apply Cor^2 above. $\exists (U, \varphi)$ st.

$\{E_i\}$ is coordinate frame.

Since $\{E_i\}$ is parallel transport of

o.n.b. $\xrightarrow[\text{isometric}]{P^\gamma \text{ is}}$ $g_p(E_i, E_j) = \delta_{ij}$. $\forall p$

i.e. $\{E_i\}$ is ortho. at any pt.

So $\varphi : U \simeq \mathbb{R}^m$