Brownian Motions

(5) Path properties of Bm:

① Area range:

Thm. (Lévy.)

$\mathcal{L}_2(B[0,1]) = 0$. a.s. where $\mathcal{L}_d$ is Lebesgue measure on $\mathbb{R}^d$. B_t on $\mathbb{R}^2$.

Pf: **Lemma.** $\mathcal{L}_2(B[0,1] \cap B[2,3]) = 0$. a.s.

Pf: Set $X = \mathcal{L}_2(B[0,1])$.

$$3\mathbb{E}(X) = \mathbb{E}(\mathcal{L}_2(B[0,3]))$$
$$\leq \sum_0^2 \mathbb{E}(\mathcal{L}_2(B[j,j+1]))$$
$$= 3\mathbb{E}(X).$$

"=" holds when $\mathcal{L}_2(B[i,i+1]) \cap \mathcal{L}_2(B[j,j+1]) = 0$. a.s.

i) Prove: $\mathbb{E}(X) < \infty$.

$X > n \Rightarrow$ Bm leaves $\overset{\sqrt{n}}{\boxed{0.}}\sqrt{n}$

So: $\mathbb{P}(X > n) \leq \mathbb{P}(B_1^* > \sqrt{n}/2)$

$$= 2\,\mathbb{P}(B_1 > \sqrt{n}/2) \leq 2e^{-n/8}.$$

ii) Set $Y = B(2) - B(1)$. $B_1(t) =: B(t)$. $0 \leq t \leq 1$.

$B_2(t) =: B(t+2) - Y$. $t \geq 0$.

Set $R(x) = \mathcal{L}_2(B_1[0,1] \cap (x + B_2[0,1]))$

$$0 \overset{\text{Lem.}}{=} \mathbb{E}(R(Y)) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-|x|^2/2}\, \mathbb{E}(R(x))\, dx$$

$\Rightarrow R(x) = 0$. $\mathcal{L}_2$-a.s.

But with Steinhaus Thm. if $B[0,1]$ has positive measure $\Rightarrow \mathcal{L}_2(\{x \in \mathbb{R}^2 \mid R(x) > 0\}) > 0$.

Cor. $\forall x, \eta \in \mathbb{R}^d$. $d \geq 2$. $\Rightarrow \mathbb{P}_x(\eta \in B[0,1]) = 0$

Pf: Project on the first 2 coordinates

We consider $d = 2$. By Fubini:

$$\int_{\mathbb{R}^2} \mathbb{P}_\eta(x \in B[0,1]) = \mathbb{E}(\mathcal{L}_2(B[0,1])) = 0$$

$$\Rightarrow \mathbb{P}_\eta(x \in B[0,1]) = \mathbb{P}_x(\eta \in B[0,1]) = 0.$$

For "$[0,1]$" can be approxi. by "$[\varepsilon, 1]$".

③ Dimension of path:

Denote: m_α is α-Hausdorff measure

Thm. For $d \geq 2$, $\Rightarrow m_2(B[0,1]) < \infty$. So:
$\dim B[0,1] \leq 2$. a.s.

Pf: Cover $B[0,1]$ by balls:

$$\{B(B(\tfrac{k}{n}),\ \max_{\frac{k}{n} \leq t \leq \frac{k+1}{n}} |B_t - B_{k/n}|)\}_0^{n-1}.$$

But $\mathbb{E}(\max_{[0,\frac{1}{n}]} |B_t|^2) = \frac{1}{n} \mathbb{E}(B_1^{*2})$

$$\Rightarrow \mathbb{E}\left(\varliminf_{n\to\infty}\sum_{0}^{n-1}\left(2\max_{[\frac{k}{n},\frac{k+1}{n}]}|B_t-B_{k/n}|^2\right)^2 \overset{\text{Fatou's}}{\lesssim}\right.$$

$$\varliminf_{n\to\infty} 2\,\mathbb{E}\left(\sum_{0}^{n-1}\square\right) \overset{\text{Markov}}{=} 2\,\mathbb{E}(B_1^{*2}) < \infty.$$

<u>Cor.</u> $m_2(B[0,1])=0$. a.s. $(\mathcal{L}_2 = m_2)$

<u>Thm.</u> (McKean's)

$A \underset{\text{closed}}{\subseteq} \mathbb{R}^{\geq 0}$. B_t is d-dim BM. Then: a.s.

$\dim B(A) = 2\dim A \wedge d$.

③ Capacity and polar sets:

<u>Def</u>: For (E, e) a metric space. with measure μ. $A \subset E$. Borel set.

i) α-potential: $\phi_\alpha(x) =: \int d\mu(y)/\ell(x,y)^\alpha$.

ii) α-energy of μ: $I_\alpha(\mu) =: \int \phi_\alpha\, d\mu$.

iii) α-capacity of (E,ℓ): $\mathrm{cap}_\alpha(E) =:$

$1/\inf\{I_\alpha(\mu) \mid \mu \text{ is p.m. on } E\}$.

If $E = \mathbb{R}^d$. $K: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{\geq 0}$. Kernel.

iv) K-energy of μ: $I_K(\mu) =: \iint K(x,y)\,d\mu(x)\,d\mu(y)$.

v) K-capacity of A: $\mathrm{cap}_K(A) =: 1/\inf\{$

$I_K(\mu) \mid \mu$ is p.m. on $A\}$.

We say A is polar for BM if: $\forall x \in \mathbb{R}^d$. $\mathbb{P}_x(B_t \in A.$ for some $t > 0) = 0$. It connects well with k-capacity.

<u>Thm.</u> (Kakutani's)

A closed set Λ is polar for d-dim. BM. $\iff$ Λ has zero k-capacity. for $k =:$

$$\begin{cases} |\log(1/|x-y|)|. & d=2 \\ |x-y|^{2-d} . & d \geq 3. \end{cases}$$

<u>Thm.</u> G is green func. Martin kernel:

$M(x,y) =: G(x,y)/G(0,y)$. $T =: \begin{cases} \tau_0 . & d=2 \\ \infty . & d \geq 3. \end{cases}$

Where $0 \in D$. bdd. $\Lambda \subset D$. Then:

$$\frac{1}{2} \mathrm{cap}_M(\Lambda) \leq \mathbb{P}_0(\exists t \in (0,T]. B_t \in \Lambda)$$
$$\leq \mathrm{cap}_M(\Lambda)$$

<u>Pf:</u> $\Lambda = \cup \Lambda_n$. where Λ_n is cpt. $d(0.\Lambda_n) > 0$

1) $d \geq 3$. Note: $G(0,y) \in (0. \infty)$.

$\Rightarrow \mathrm{cap}_M(\Lambda) \asymp \mathrm{cap}_K(\Lambda)$.

2) $d = 2$. $G(x,y) = -\frac{1}{\pi}\log|x-y| + \mathbb{E}_x(\frac{1}{\pi}\log|B_\tau - y|)$

$\Rightarrow \log|x-y|$ decides the finiteness of Martin energy. ($T = \tau_{B_R}$. $\Lambda \subset B_R$)

④ Multiple points:

For $(B_k(t))_1^p$ indept BMs. we care about: if $\exists (t_k)$. st. $B_1(t_1) = \cdots = B_p(t_p)$. intersections.

Lemma. $A \underset{\text{close}}{\subset} \mathbb{R}^d$. $\Rightarrow \dim A = \sup\{\alpha \mid cap_\alpha(A) > 0\}$.

Lemma. (Energy Method)

For $m_\alpha = \lim_{\varepsilon \downarrow 0} m_\alpha^\varepsilon$. def of Hausdorff measure. If $\alpha \geq 0$. $\mu \neq 0$ on metric space E. Then: $\forall \varepsilon > 0$. we have:

$$m_\alpha^\varepsilon(E) \geq \mu(E)^2 \Big/ \int_{\rho(x,y)<\varepsilon} d\mu(x)\,d\mu(y) / \rho^\alpha(x,y).$$

Cor. $m_\alpha(E) < \infty \Rightarrow I_\alpha(\mu) = \infty$.

Pf: For $\delta > 0$. choose covering (A_n). of E. st. $\sum_n d(A_n)^\alpha \leq m_\alpha^\varepsilon(E) + \delta$. $d(A_n) < \varepsilon$.

$$\mu(E)^2 \leq \left(\sum \mu(A_n)\right)^2 \leq \sum d(A_n)^\alpha \sum \frac{\mu(A_n)^2}{d(A_n)^\alpha}$$

$$\leq (m_\alpha^\varepsilon(E) + \delta)\left(\sum \int_{A_n \times A_n} \frac{d\mu(x)\,d\mu(y)}{\rho(x,y)^\alpha}\right)$$

$$\leq (m_\alpha^\varepsilon(E) + \delta)\left(\int_{\rho(x,y)<\varepsilon} \square\right)$$

Thm. i) For $d \geq 4$. a.s. two BMs won't have intersection points. except common starting point.

ii) For $d \leq 3$. a.s. the intersection of two BMs is nontrivial. containing more than B_0.

Pf: 1') $d \geq 5$. since $\dim B_1[0,\infty) = 2$.

So $(d-2)$-capacity is zero. by Lemma.

2') $d = 4$. since $m_2(B_1[0,\infty)) < \infty$

So: $\mathrm{cap}_2(B_1[0,\infty)) = 0$. by energy method.

3') $d \leq 3$. (Only consider $d=3$. by proj.)

$\mathrm{cap}_1(B_1[0,\infty)) > 0 \Rightarrow \mathbb{P}(\bigcup_n (B_1[n,n+1] \cap B_2 \cdots)) > 0$

So: $\exists\, n_0$ s.t. $\mathbb{P}(B_1[n_0,n_0+1] \cap B_2[n_0,n_0+1]) > 0$

By Markov property. $\mathbb{P}(B_1 \cap B_2 \neq \emptyset$ in $\cdots$ i.o.$) = 1$

$\Rightarrow \mathbb{P}(B_1 \cap B_2) \geq \mathbb{P}(\cdots$ i.o.$) = 1$.

Thm. (Multiple case)

i) For $d \geq 3$. a.s. three indept BMs in $\mathbb{R}^d$. have empty intersection. except common starting.

ii) For $d = 2$. a.s. finite p indept BMs in $\mathbb{R}^d$. have nontrivial intersection. more than starting.

Pf: i) For $d = 3$:

Lemma. $\forall\, \Lambda \subseteq_{cpt} \mathbb{R}^3$. B_i. $i=1,2$ indept BMs in $\mathbb{R}^3$. $B_1(0)$. $B_2(0) \in \Lambda$.

$\Rightarrow m_1(B_1[0,\infty) \cap B_2[0,\infty) \cap \Lambda) < \infty$.

Pf: By def of m_α and Fatou's

$\Rightarrow$ By energy method. $\mathrm{cap}_1(B_1 \cap B_2 \cap \Lambda) = 0$

For ii). we need a co-dimension approach:

take a suitable random set Θ. check: $\mathbb{P}(\Theta \cap A) \neq 0$

Def: $C \underset{cpt}{\subset} \mathbb{R}^d$. unit cube. $\mathcal{C}_n$ is collection of dyadic subcubes of C with length 2^{-n}.

Given $\gamma \in [0,d]$. set $G_1 = \{$ cubes in $\mathcal{C}_1$ are kept with $p = 2^{-\gamma}$ indeptly $\}$. and $S(1)$ is its union. Inductively:

$G_{n+1} =: \{$ cubes $\subset S(n)$ in $\mathcal{C}_{n+1}$ are kept with $p = 2^{-\gamma}$ indeptly $\}$. with $S(n+1)$ union.

$\Rightarrow$ Random set $I(\gamma) =: \bigcap_{n \geq 1} S(n)$ is called percolation limit.

Thm. (Hawkes)

For $\forall \gamma \in [0,d]$. $A \subset C$. closed

i) $\dim A < \gamma \Rightarrow A \cap I(\gamma) = \emptyset$. a.s.

ii) $\dim A > \gamma \Rightarrow A \cap I(\gamma) \neq \emptyset$ with prob. > 0.

Thm. For $d = 2.3$. B_1. B_2 are indept BMs in $\mathbb{R}^d$.

$\Rightarrow$ a.s. $\dim(B_1[0,\infty) \cap B_2[0,\infty)) = 4-d$.

Pf: For $d=3$. ($d=2$. similarly).

Set $\gamma < 1$. $\beta > 1$. $\gamma + \beta < 2$. Therefore:

$\mathbb{P}(B_1[0,\infty) \cap (I(\gamma) \cap I(\beta))) > 0$.

$\Rightarrow \dim(B_1[0,\infty) \cap I(\gamma)) \geq \beta$. w.p. > 0.

So: $\mathbb{P}(\mathrm{cap}_\beta(B_1[0,\infty) \cap I(y)) > 0) > 0$.

$\overset{\beta>1}{\Rightarrow}$ $\mathbb{P}(B_1[0,\infty) \cap B_2[0,\infty) \cap I(y) \neq \emptyset) > 0$

$\Rightarrow$ $\dim(B_1[0,\infty) \cap B_2[0,\infty)) \geq \gamma$. w.p. > 0.

Note it's equi.: $\mathbb{P}(\bigcap_{n\geq 1} \lim_{\varepsilon \downarrow 0} \inf\{\sum_{k\geq n} d(E_k)^\gamma$

$| \cup E_k$ is a covering. $d(E_k) < \varepsilon\} = \infty) > 0$.

By Kolmogorov 0-1 law $\Rightarrow$ $\mathbb{P}(\dim(\cdots) \geq \gamma) = 1$.

Then: set $\gamma \to 1^-$.

Converse is from lemma: $M_1(B_1 \cap B_2 \cap \Lambda) < \infty$.

Cor. For $d=2$. $\forall p \in \mathbb{Z}^+$. $\dim(B_1[0,\infty) \cap$

$\cdots B_p[0,\infty)) = 2$.

Pf: As above. by induction on p.

Rmk: We have concluded ii) of Thm.

Thm. (B_t) is 2-dimension BM. Then. a.s.

$\exists x \in \mathbb{R}^2$. st. $T(x) = \{t \geq 0 \mid B_t = x\}$ is uncountable. But $\dim T(x) = 0$. for all such points.

Thm. (large occupation-time)

B_t is 2-dim BM. $D = B(0,1)$. $\mu(A) =: \int_0^{\tau_D} I_{\{B_s \in A\}} ds$

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^2} \mu(B(x,\varepsilon)) / \varepsilon^2 \log \tfrac{1}{\varepsilon} = \sup_x \overline{\lim_{\varepsilon \downarrow 0}} \frac{\mu(B(x,\varepsilon))}{\varepsilon^2 \log 1/\varepsilon} = 2, \text{ a.s.}$$