Sample Path Generation

(1) <u>Brownian motions:</u>

We can't sample its full path $(B_t)_{[0,T]}$ but only skeleton $(B_{t_1}, \cdots B_{t_n})$. $B_t \in \mathbb{R}^1$.

<u>Rmk:</u> We can interpolate on the skeleton to get approxi. path. But it'll not be adaptive.

① <u>Cholesky decomp.:</u>

Note $(B_{t_1}, \cdots B_{t_n}) \sim N(0, \Sigma)$. $\Sigma_{ij} = t_i \wedge t_j$.

Let $\Sigma = AA^T$, $X = (X_1 \cdots X_n)$. $X_i \overset{i.i.d}{\sim} N(0,1)$

$\Rightarrow AX \sim (B_{t_1} \cdots B_{t_n})$.

We can compute $A = \begin{pmatrix} \sqrt{t_1} & 0 & \cdots & 0 \\ \vdots & \sqrt{t_2 - t_1} & 0 & \vdots \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \cdots & \sqrt{t_n - t_{n-1}} \end{pmatrix}$

② <u>Random Walk approach:</u>

Set $\Delta B_k = B_{t_k} - B_{t_{k-1}}$. $\Delta B_1 = B_{t_1}$. Then we have indept increments. And $\Delta B_k = \sqrt{\Delta t_k}\, X_k$.

Where $\Delta t_k = t_k - t_{k-1}$. $\Delta t_1 = t_1$. $X_k \overset{i.i.d.}{\sim} N(0,1)$

$\Rightarrow B_{t_k} = \sum_1^k \Delta B_i$

Rmk: It's equi. with Cholesky decomp.

③ Brownian bridge:

It bases on a property of BM:

$$B_s \mid B_u = x,\ B_t = y \sim N\left(\frac{(t-s)x + (s-u)y}{t-u},\ \frac{(s-u)(t-s)}{t-u}\right)$$

for $u < s < t$.

1) First sample B_{t_n} ($t_n = T$)

2) let k_1 s.t. t_{k_1} is closest to $T/2$

$\Rightarrow$ sample $B_{t_{k_1}}$ under B_{t_n} and $B_0 = 0$.

3) sample $B_{t_{k_2}}$, $B_{t_{k_3}}$. s.t. $t_{k_2} \approx \frac{T}{4}$. $t_{k_3} \approx \frac{3T}{4}$ by the cond. dist.

Rmk: i) We can still represent $(B_{t_1} \cdots B_{t_n})$ in func. of $(X_k) \overset{i.i.d.}{\sim} N(0,1)$. But it's not equi. with ①. ② now.

ii) It can be seen as dimension-reduction technique since its construction starts from coarse one to finer one. And most of quantity interests in real life only depend on coarse structure

④ Karhunen-Loève expansion:

Thm. (Mercer's)
$K \in C(D^2; \mathbb{R}^1)$, $D \subset \mathbb{R}^1$. Set $K[\cdot]: L^2 \to L^2$, $f \mapsto \int_D K(x,y) f(y)\, dy$. HS op. If (λ_i), (e_i) are eigenvalues and eigenvectors of $K[\cdot]$. Then:
$K(s,t) = \sum_i \lambda_i e_i(t) e_i(s)$ in sense of uniform and absolute.

For (X_t) stochastic process. $R_X(s,t)$ is its cov. func.

Lem. (X_t) is L^2-conti. $\Leftrightarrow$ $R_X \in C(\mathbb{R}^2)$.

Next. assume X_t is centered, L^2-conti.

Lem. Let $K = R_X$. $\Rightarrow$ $K[\cdot] \in \mathcal{K}(L^2(D))$ and

Self-adjoint. positive definite.

Recall Self-adjoint. cpt operator K has complete ONS (e_i) of $L^2(D)$. st. (e_i) is its eigenvector.

$\Rightarrow X_t \overset{L^2}{=} \sum_1^\infty X_i e_i(t)$. where $X_i = \int_D X_t e_i(t) dt$.

Lem. i) $\mathbb{E}(X_i) = 0$ ii) $\mathbb{E}(X_i X_j) = \lambda_j \delta_{ij}$

Pf: i) is from Fubini Thm.

ii) $LHS = \int_D \int_D \mathbb{E}(X_s X_t) e_i(s) e_j(t)$

$= \int_D K[e_i] e_j = \langle Ke_i, e_j \rangle$

$= \lambda_i \delta_{ij}$.

Set (φ_i) is o.n.b from $(s \wedge t)[\cdot]$. i.e.

$\varphi_i = e_i / \lambda_i^{\frac{1}{2}}$. For $X_t = B_t$:

$B_t = \sum \sqrt{\lambda_i} \varphi_i(t) z_i$. where $z_i \overset{i.i.d}{\sim} N(0,1)$

since $\mathbb{E}(\varphi_i^2) = \langle Ke_i, e_i \rangle / \lambda_i = 1$.

$\mathbb{E}(\varphi_i \varphi_j) = 0$. $\mathbb{E}(\varphi_i) = 0$.

Rmk: In fact, $\lambda_i = \left(\frac{2}{(2i+1)\pi}\right)^2$. $\varphi_i = \sqrt{2} \sin\left(\frac{(2i+1)\pi t}{2}\right)$

⑤ Wavelet construction:

Consider $\psi(t) = I_{[0,\frac{1}{2})} - I_{[\frac{1}{2},1)}$

Def: $\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$ Haar basis

and $X_0, X_{n,k} \overset{i.i.d.}{\sim} N(0,1)$.

We have: $B_t = X_0 t + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} X_{n,k}\varphi_{n,k}(t)$. st.

$\varphi_{n,k}(t) = \int_0^t \psi_{n,k}(s)ds = 2^{-n/2}\varphi(2^n t - k)$, $\varphi(t) = \int_0^t \psi(s)ds$.

Rmk: It's interpreted as: $(\varphi_{n,k})_k$ describe

the macroscopic behavior of BM

and $(\varphi_{n,k})_n$ provides finer solution.

Set $B_t^{(N)} = X_0 t + \sum_{n=0}^{N}\sum_{k=0}^{2^n-1} X_{n,k}\varphi_{n,k}(t)$.

prop. $B_t^{(N)} = B_t$ on $D^{(N)} = \{k/2^{N+1}\}_{0\le k\le 2^{N+1}}$.

(2) Lévy process:

Note Lévy process is infinite divisible

so we can use Random Walk approach. e.g. normal inverse Gaussian

But not every Lévy process is applicable.

① Compound Poisson:

Recall Lévy process can be decomposed into three parts. If the jump part has only finite jumps in cpt interval. then it has finite activity and is compound Poisson process:

$Z_t = Z_0 + \sum_1^{N_t} X_k$. N_t Poi. X_k. i.i.d. jumps.

i) Sample value of N_t:

a) Note $(N_{t_1}, \cdots N_{t_n})$ also satisfies:

$N_{t_k} - N_{t_{k-1}} \sim Poi(\lambda(t_k - t_{k-1}))$. indept.

And poisson dist. can be simulated by inverse method.

b) Poisson bridge: $N_s | N_t = n \sim Bi(\frac{s}{t}, n)$

ii) Sample trajectory of N_t:

For point seq. $(T_1, T_2 \cdots T_{N_t})$:

a) Note $T_k - T_{k-1} \overset{i.i.d.}{\sim} exp(\lambda)$. And $exp(\lambda)$ can be obtained by the inverse method.

b) $(T_1 \cdots T_{N_t}) | N_t = n \sim$ order stat.

first sample $(t_{U_1}, \cdots t_{U_n})$. $U_k \overset{i.i.d.}{\sim}$ $U[0,1]$ and order them.

e.g. Consider $dS_t = \mu S_{t-} dt + \sigma S_{t-} dB_t + S_{t-} dJ_t$.

where $J_t = \sum_1^{N_t} (X_j - 1)$. $X_j \geq 0$. indept. N_t indept of X, B. We have:

$S_t = S_{T_n} \exp(\sigma(B_t - B_{T_n}) + (\mu - \frac{\sigma^2}{2})(t - T_n))$

for $T_n \leq t < T_{n+1}$.

$S_t - S_{t-} = S_{t-}(X_{n+1} - 1)$ for $t = T_{n+1}$.

So: $S_{T_n} = X_n S_{T_n-}$. Then:

$S_t = S_0 \exp(\sigma B_t + (\mu - \frac{\sigma^2}{2})t) \prod_1^{N_t} X_j$.

$\Rightarrow$ sample N_t, B_t, then $(X_1, \cdots X_{N_t})$.

② Variance Gamma model:

Exponential Lévy process is $S_t = S_0 e^{Z_t}$ where Z_t is Lévy process. And var. gamma model is pure-jump exp. Lévy process where $Z_t = U_t - D_t$. U, D are two indept gamma process. (Lévy process with gamma dist. increment. i.e.

$X_{s,t} \sim \Gamma_{(k(t-s)), \theta}$. $k > 0$)

Rmk: Z_t has infinite activity.

We can still sample the increments of gamma process, which's gamma dist.

$\Rightarrow$ Apply acceptance - rejection method.

Note that a) $X \sim \Gamma_{k,1} \Rightarrow \theta X \sim \Gamma_{k,\theta}$.

b) $\Gamma_{k,1} \to N(0,1)$ ($k \to \infty$)

c) $\Gamma_{k,1}$ has fatter tail than $N(0,1)$ and thinner than exponential dist.

$\Rightarrow g$ is convex combination of densities of normal dist. and exponential dist.

Rmk: For scale, shape para. θ_u, θ_D ; k_u, k_D of gamma process U, D. If:

$k_u = k_D = 1/\theta$. then $Z_t \hat{=} U_t - D_t = W_{G_t}$

for G_t is $(\theta, 1/\theta)$ - gamma process and $W_t = \mu t + \sigma B_t$. BM with $\mu = \theta^{-1}(\theta_u - \theta_D)$. $\sigma^2 = 2\theta_u\theta_D/\theta$. B_t. SBM indept of G_t.

So to sample Z_t. $\Leftrightarrow$ sample one BM and one gamma process now.

<u>Rmk</u>: This is motivation of param of Var. gamma process:

since $Z_t \mid G_t \sim N(0, G_t)$.

③ <u>Approxi. of Lévy process:</u>

We consider the case: infinite activity. and has charac. triplet: $(\gamma, 0, \nu)$.

$\Rightarrow Z_t = \gamma t + \sum_{0<s\le t} \Delta Z_s I_{|\Delta Z_s|\ge 1} + \lim_{\varepsilon \to 0} N_t^{\varepsilon}$.

where $N_t^{\varepsilon} = \sum_{0<s\le t} \Delta Z_s I_{\varepsilon\le|\Delta Z_s|<1} - t\int_{\varepsilon\le|z|<1} z\,\nu(dz)$

Approxi. Z_t by $Z_t^{\varepsilon} = \gamma t + \sum_{0<s\le t} \Delta Z_s I_{|\Delta Z_s|\ge 1} + N_t^{\varepsilon}$

So now Z_t^{ε} is compound Poisson process with drift γt $\Rightarrow$ can be simulated.

<u>Thm.</u> For $\operatorname{Var}(Z_t - Z_t^{\varepsilon}) = t\int_{|z|<\varepsilon} z^2 \nu(dz) \stackrel{\Delta}{=} t\,\sigma(\varepsilon)^2$.

ii) $\forall f \in C^1$. s.t. $|f'| \le c$. We have:

$$|\mathbb{E}(f(Z_t)) - \mathbb{E}(f(Z_t^\varepsilon))| \le C\,\sigma(\varepsilon)\,t^{\frac{1}{2}}.$$

iii) $\sigma(\varepsilon)^{-1}(Z - Z_\varepsilon) \xrightarrow{\mathcal{L}} N(0,1) \iff \frac{\sigma(\varepsilon)}{\varepsilon} \xrightarrow{\varepsilon\to 0} \infty$.

<u>Rmk</u>: So $Z_t \approx Z_t^\varepsilon + \sigma(\varepsilon) B_t$.

<u>e.g.</u> In case of gamma process.

We have $\sigma(\varepsilon) \sim \varepsilon$. It means

approxi. by Z^ε is very good.