Discretization of SDEs

<u>Thm</u> Z_t is r-dim càdlàg semimart. with $Z_0 = 0$. $F: \mathbb{R}^{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is Lip. i.e.:

$|F(t,x) - F(t,y)| \leq k_t |x-y|$. $\forall x, y \in \mathbb{R}^d$.

Then: $X_t = X_0 + \int_0^t F(s, X_{s-}) dZ_s$ admits a unique sol. X_t is semimart.

<u>Rmk</u>: For $F \in C^2(\mathbb{R}^{\geq 0} \times \mathbb{R}^n)$, $x \mapsto F(t,x)$ and $x \mapsto DF_i(t,x) F_i(t,x)$ are Lip. for $\forall t \in \mathbb{R}^{\geq 0}$, $\forall 1 \leq i \leq d$. Where $F_i(t,x) = (F(t,x)_i^j)_{j=1}^n$. Then the result also works for Stratonovich case: $X_t = X_0 + \int_0^t F(s, X_{s-}) \circ dZ_s$

(1) <u>Euler method</u>.

Consider $dX_t = V_0(X_t) dt + \sum_1^d V_i(X_t) dB_t^i$. St.

V_i's are uniform Lip. and linearly bdd.

So its sol. X_t exists uniquely.

Set grid $D = \{0 = t_0 < t_1 \cdots < t_N = T\}$. $|D| = \max |\Delta t_k|$. $\Delta t_k = t_k - t_{k-1}$. $\lfloor t \rfloor = \sup\{t_i \mid t_i \leq t\}$.

Def: For approxi. $\bar{X}^D$ for X.

i) $\bar{X}^D \to X$ strongly if $\lim_{|D|\to 0} \mathbb{E}|X_T - \bar{X}^D_T| = 0$

And it has strong order γ if

$\mathbb{E}(|X_T - \bar{X}^D_T|) \le C|D|^\gamma$. $(|D|\to 0)$

ii) For class $\mathcal{G}$ of func. $f: \mathbb{R}^n \to \mathbb{R}^1$.

$\bar{X}^D \to X$ weakly w.r.t $\mathcal{G}$ if $\forall f \in \mathcal{G}$

$\lim_{|D|\to 0} \mathbb{E}(f(\bar{X}^D_T)) = \mathbb{E}(f(X_T))$.

It has weak order $\gamma > 0$ if $\forall f \in \mathcal{G}$. $|\mathbb{E}(f(\bar{X}^D_T)) - \mathbb{E}(f(X_T))| \le C|D|^\gamma$.

Rmk: For $\mathcal{G} = Lip.$: Strong $\Rightarrow$ weak.

① Euler - Maruyama method:

Note for ODE $x'(t) = V(x(t))$. $X(0) = X_0$.

$X(t_i) = X(t_{i-1}) + X'(t_{i-1})\Delta t_i + O(\Delta t_i^2)$

$= X(t_{i-1}) + V(X(t_{i-1}))\Delta t_i + O(\Delta t_i^2)$

Set $\bar{X}_{t_i} = V(\bar{X}_{t_{i-1}})\Delta t_i + \bar{X}_{t_{i-1}}$. $\bar{X}_{t_0} = X_0$.

So $|X(T) - \bar{X}_{t_N}| \le \sum_i^N O(\Delta t_i^2) \le T\,O(|D|)$.

As for SDE:

Set $\bar{X}_{t_i} = \bar{X}_{t_{i-1}} + V_0(\bar{X}_{t_{i-1}})\Delta t_i + \sum_1^d V_j(\bar{X}_{t_{i-1}})\Delta B^j_{t_i}$

extend it to whole path by interpolate:

$\bar{X}_t = \bar{X}_{\lfloor t \rfloor} + V(\bar{X}_{\lfloor t \rfloor})(t - \lfloor t \rfloor) + \sum_1^d V_i(\bar{X}_{\lfloor t \rfloor})(B^i_t - B^i_{\lfloor t \rfloor})$

Rmk: Now the scheme may not also have order one. since $B_t \sim \sqrt{t}$

Thm. Under $X(0)=x$. Approxi. $\bar{X}_t$ satisfies:

$\mathbb{E}(\sup_{0 \le t \le T} |X_t - \bar{X}_t|) \le C|D|^{\frac{1}{2}}$.

Rmk: So it has strong order $1/2$.

Pf: Denote $B^0_t = t$ and recall:

$\mathbb{E}(\sup_{0 \le t \le T} |X_t|^2) \le C(1+|x|^2)$.

Next estimate $e_t = \mathbb{E}(\sup_{0 \le s \le t} |X_s - \bar{X}_s|^2)$

$X_s - \bar{X}_s = \sum_0^d \int_0^s (V_i(X_u) - V_i(\bar{X}_{\lfloor u \rfloor}))\, dB^i_u$.

$= \sum_0^d \Big(\int_0^s (V_i(X_u) - V_i(X_{\lfloor u \rfloor})) +$

$(V_i(X_{\lfloor u \rfloor}) - V_i(\bar{X}_{\lfloor u \rfloor}))\Big)\, dB^i_u$

So: $e_t \lesssim \sum_0^d \mathbb{E}(\sup_{0 \le s \le t} |\int_0^s V_i(X_u) - V_i(X_{\lfloor u \rfloor})\, dB^i_u|^2) + \mathbb{E}(\sup_{0 \le s \le t} |\int_0^s V_i(X_{\lfloor u \rfloor}) - V_i(\bar{X}_{\lfloor u \rfloor})\, dB^i_u|^2) =: \sum_0^d C^i_t + d^i_t$.

Using Doob's inequi. and Hölder inequi:

$d^0_t \le K^2 T \int_0^t e_s ds$. $d^i_t \le 4K^2 \int_0^t e_s ds$. $1 \le i \le d$.

where K is common Lip const. of V_i.

$$C^0_t \overset{\text{Hölder}}{\le} K^2 T \int_0^t \mathbb{E}\left(\left|\sum_0^d \int_{[u]}^u V_i(X_s)\, dB^i_s\right|^2\right) du$$

$$\le K^2 T (d+1) \int_0^t \sum_0^d \mathbb{E}\left(\left|\int_{[u]}^u V_i(X_s)\, dB^i_s\right|^2\right) du$$

$$\underset{\text{Linear growth}}{\overset{\text{Hölder}}{\le}} CK^2 \int_0^t (u-[u])\left(\int_{[u]}^u \mathbb{E}(1+|X_s|^2)\,ds + d\int_{[u]}^u \mathbb{E}(1+|X_s|^2)\,ds\right) du$$

$$\le C(1+X^2) \int_0^t (u-[u])^2 + d(u-[u])\, du.$$

$$\le T\, C\, (1+X^2)(|D|+d)|D|.$$

With Doob's inequi. we also have:

$C^i_t \le C(1+|x|^2)|D|$. $1 \le i \le d$.

So: $e_t \le C|D| + C\int_0^t e_s ds$.

Apply Gronwall's inequi.: $e_t \le C|D|$.

Apply Hölder again to get the result

Rmk: It can be used to prove the uniqueness and existence of SDE:

Note first we also have:

$\mathbb{E}(\sup_{0\le t\le T} |\bar{X}_t^D|^2) \le C(1+|x|^2)$.

1) Uniqueness is obvious.

2) Note: $\mathbb{E}(\sup_{0\le t\le T} |\bar{X}_t^D - \bar{X}_t^{D'}|^2) \le C|D| \vee |D'|$.

3) Set $\tilde{X} = \lim \bar{X}^D$. prove $\tilde{X}$ also satisfies the SDE since $\bar{X}^D$'s satisfied it.

Next, we consider the weak convergence of the Euler method.

<u>Rmk</u>: Weak approxi. of SDE can be used as a numerical method to solve the linear parabolic PDE:

Since $u(t,x) = \mathbb{E}(f(X_T)|X_t = x)$ solve the Kolmogorov backward eq.:

$$\begin{cases} \partial u/\partial t = -Lu \\ u(T,x) = f(x) \end{cases}$$ where $L = \tilde{V}_0 + \frac{1}{2}\sum_i V_i^2$.

$\tilde{V}_0 = V_0 - \frac{1}{2}\sum DV_i \cdot V_i$,

and $Vg(s) = \nabla g(s) \cdot V(s)$.

<u>Thm</u>. If V_i's are C^∞-bdd. $g = C^\infty \cap \varepsilon f$ is

poly. desc}. $h := \Delta t_i = T/n$. $\forall i$. Then:

$$e(T,h,f) \overset{\Delta}{=} \mathbb{E}(f(\bar{X}_T^0) - f(X_T))$$

$$= h\int_0^T \mathbb{E}(\psi_1(s,X_s))ds + h^2 C_2(T,f) + O(h^3)$$

where ψ_1 is given by

$$\psi_1(t,x) = \frac{1}{2}\sum_{i,j=1}^n V^i(x)V^j(x)\partial_{(i,j)}u(t,x) + \frac{1}{2}\sum_{i,j,k=1}^n V^i(x)a_k^j(x)\partial_{(i,j,k)}u(t,x)+$$
$$+\frac{1}{8}\sum_{i,j,k,l=1}^n a_j^i(x)a_l^k(x)\partial_{(i,j,k,l)}u(t,x) + \frac{1}{2}\frac{\partial^2}{\partial t^2}u(t,x)+$$
$$+\sum_{i=1}^n V^i(x)\frac{\partial}{\partial t}\partial_i u(t,x) + \frac{1}{2}\sum_{i,j=1}^d a_j^i(x)\frac{\partial}{\partial t}\partial_{(i,j)}u(t,x),$$

where $\partial_I = \frac{\partial^k}{\partial x^{i_1}\cdots\partial x^{i_k}}$ for a multi-index $I = (i_1,\ldots,i_k)$ and $a_j^i(x) = \sum_{k=1}^d V_k^i(x)V_k^j(x)$, $1 \leqslant i,j \leqslant n$.

Lem. Under cond. above. Sol. $u(t,x)$ of PDE above is smooth and all its derivatives are poly-growth.

Pf: By Fubini. Thm.

Lem. Under cond. above. We have:

$$\mathbb{E}(u(t_{i+1},\bar{X}_{i+1}) \mid \bar{X}_i = x) = u(t_i,x) + h^2\psi_1(t_i,x) + O(h^3)$$

Pf: WLOG. let $i=0$.

Apply Taylor expansion on $u(h, x+\Delta x)$ on h and ΔX:

$$u(h, x+\Delta x) = u(0,x) + h\,\partial_t u(0,x) + \frac{1}{2}\cdot h^2\partial_t u(0,x) + h\sum \Delta X^i\,\partial_t\partial_i u(0,x) +$$

$\frac{1}{2}h\,\Sigma_{i,j}\,\Delta X^i \Delta X^j\, \partial_t \partial_{(i,j)} u(0,x) + \sum_k^4 \sum_{i_k=1}^n \Delta X^{i_1} \dots$

$\Delta X^{i_k}\, \partial_{(i_1 \dots i_k)} u(0,x) + O(h\Delta X^3) + O(\Delta X^5)$

Insert $\Delta X = \Delta \bar{X} = V_0(x)h + \sum_1^r V_i(x)\Delta B_h^i$ and take expectation on both sides.

Combine with $\Delta B_h^i \sim \sqrt{h}$ and eq.:

$\partial_t u(0,x) = -L u(0,x)$.

<u>Pf</u>: Note $e(T,h,f) = \mathbb{E}(u(T,\bar{X}_N^0) - u(0,x))$

$= \sum_0^{N-1} \mathbb{E}(u(t_{i+1},\bar{X}_{i+1}) - u(t_i,\bar{X}_i))$

$= \sum_0^{N-1} h^2\, \mathbb{E}(\psi_1(t_i,\bar{X}_i)) + O(h^3)$

$\leq C h^2 N + N\,O(h^3) = CT(h + O(h^2))$

Since by cond.: $|\mathbb{E}(\psi_1(t,x))| \leq C$.

Also: $|h \sum_0^{N-1} \mathbb{E}(\psi_1(t_i,\bar{X}_i)) - \int_0^T \mathbb{E}(\psi_1(t,X_t))|$

$\leq h \sum_i |\mathbb{E}(\psi_1(t_i,\bar{X}_i) - \psi_1(t_i,X_{t_i}))| +$

$|h \sum_i \mathbb{E}(\psi_1(t_i,X_{t_i})) - \int_0^T \mathbb{E}(\psi_1(t,X_t))|$

$:= A + B$.

$B = O(h)$. Besides, $\psi_1 \in C^1 \Rightarrow \psi_1 \in G$. So:

$|\mathbb{E}(\psi_1(t_i,\bar{X}_i) - \psi_1(t_i,X_{t_i}))| = O(h)$.

Rmk: Note we only use scaling prop. of BM rather other gaussian prop. So we can use general Lévy process $Y_h \sim \sqrt{h}$.

Thm: If f, V_i's $\in C^4$ and has poly.-bdd deri. Then: it has weak order one.

Thm. If V_i's $\in C^\infty$ satisfies Hormander's cond. and their deri.'s are bdd.

Then: $\forall f$. bdd. measur. it has weak order one.

Rmk: It's a trading on regularity between f and V_i's.

Sometimes the Euler scheme can still work in n.s. smooth case

③ Euler - Monte - Carlo method:

Note we can approxi. $\mathbb{E}(f(X_T))$ by $\mathbb{E}(f(\bar{X}_{T^n}))$. So next step we want to approxi. the integral $\mathbb{E}(f(\bar{X}_{T^n}))$.

Rmk: Assume the SDE is driven by d-dim Lévy process. $\Rightarrow \bar{X}_N$ is func. of $(\Delta Z_n^i : 1 \leq i \leq d, 1 \leq n \leq N)$. So $\mathbb{E}(f(\bar{X}_N))$ is integral on $\mathbb{R}^{Nd}$. which still cause dim. problem if N large.

$$\Rightarrow \text{Error} = |\mathbb{E}(f(X_T)) - M^{-1} \sum_1^m f(\bar{X}_N^{(i)})|$$

$$\leq |\mathbb{E}(f(X_T)) - \mathbb{E}(f(\bar{X}_N))| + |\mathbb{E}(f(\bar{X}_N)) - M^{-1} \sum_1^m f(\bar{X}_N^{(i)})|$$

$$\triangleq \text{error}_{\text{discret}}(N) + \text{error}_{\text{int}}(M).$$

It left us two para. M, N to choose:

$\text{error}_{\text{discret}}(N) \leq C_p N^{-p}$. $p \in [\frac{1}{2}, 1)$.

$\text{error}_{\text{int}}(M) \leq C_2 M^{-2}$. $2 \in [\frac{1}{2}, 1-\varepsilon]$.

Note the computational work will be propositional to MN

So we want to optimize:

$$\min \{MN \mid C_2 M^{-2} + C_p N^{-p} \leq \varepsilon\}$$

Lagrangian $F(M, N, \lambda) = MN + \lambda (C_2 M^{-2} + C_p N^{-p} - \varepsilon)$

Let $\partial F / \partial N = \partial F / \partial M = 0 \Rightarrow M \simeq N^{p/2}$.

$\Rightarrow m \approx \varepsilon^{-1/2}$. $N \approx \varepsilon^{-1/p}$. $mN \approx \varepsilon^{-(1/p+1/2)}$.

Problem description	p	q	$M(N)$	k
Euler (Lipschitz) + MC	1/2	1/2	N	4
Euler (Lipschitz) + QMC	1/2	$1-\delta$	$N^{1/2+\delta}$	$3+\delta$
Euler (regular) + MC	1	1/2	N^2	3
Order p + MC	p	1/2	N^{2p}	$2+1/p$

$k = 1/p + 1/2$ is complexity of EMC.

Rmk: Note higher order method can't improve the cost significantly if combined with a low order scheme.

Next, consider Lévy noise case:

$dX_t = \sum_1^d V_i(X_{t-})\, dZ^i_t$.

For the Euler scheme, we can replace $B_{t_i,t_{i+1}}$ by $Z_{t_i,t_{i+1}}$ directly. If we can sample increments of Z_t. Then:

Thm. If $(V_i), f \in C_B^4$. Then its Euler scheme $\rightarrow$ sol. of SDE weakly.

Moreover, if Lévy measure ν of Z_t has bdd moments up to 8, then, it has weak rate one.

Rmk: i) The rate is smaller than BM case (even no such rate)

ii) In the proof: Z_t is approxi. by Z_t^m (exclude the jumps $> m$) so the error estimate will base on $|D|$ and m. And if ν admits high order moments, then this step can be avoided.

But generally, given Lévy-(a, Σ, ν). We don't know its dist. so can't sample it. We can still approxi. incre. ΔZ_t by compound Poi-Process ΔZ_t^ε.

Thm. If i) SDE sol. X_t and i.i.d. r.v.'s $(\zeta_i^h)_i$ satisfy: $\forall g \in C_B^4$. $\exists K_h$ const. of h

$$|\mathbb{E}(g(\zeta_i^h)) - \mathbb{E}(g(Z_h))| \le K h K_h \|g\|_{C^4}.$$

ii) $(V_i) \in C_B^4$. Z has finite moments up to order eight.

Then Euler scheme for partition D

$= \{kh\}_{k=0}^{n}$: $\bar{X}_0 = X_0$. $\bar{X}_{n+1} = \bar{X}_n + \sum_i V_i(\bar{X}_n) \xi_{n+1}^{h}$

satisfies: $\forall f \in C_b^4$.

$|\mathbb{E}(f(X_T)) - \mathbb{E}(f(\bar{X}_N))| \le C \max(u_h, h) \|f\|_{C_b^4}$

Rmk: i) Note if we can simulate Z, then we can choose $u_h = 0$.

ii) To obtain approxi. ξ_i^h of Z, we can choose $\xi_i^h = Z^{\varepsilon}_{(i-1)h,\, ih}$. So u_h will also depend on ε.

Lem. If $\lambda_\varepsilon := \nu[|z| > \varepsilon] \le C/\varepsilon^{Y}$. $\exists Y \in [0, 2]$. $\forall \varepsilon \le 1$. Then: $u_h = \varepsilon^{3-Y}$.

Note by integrability cond. on ν of Z_t. it always satisfies for $Y = 2$. So to get $u_h \sim h$: we let $\varepsilon \sim h^{1/(3-Y)}$.

(2) Advanced methods:

① Multilevel MC simulation:

Next, we consider a simulation of schemes at different time grids. i.e. set $h_1 > \cdots > h_L$ time increments. Sol. X_t will be approxi. by $\bar{X}^{(h_L)}$. And $\mathbb{E}(f(X_T))$ will base on samples of $(\bar{X}^{(h_k)})_k$.

Rmk: i) The discret. error is given by the finest discretization $\bar{X}^{(h_L)}$. And the computational cost will be average of the works w.r.t different h_k.

ii) The idea of MMC is based on the variance reduction. Note $\bar{X}^{(h_k)}$ and $\bar{X}^{(h_{k+1})}$ should be close. So: $\mathrm{Cov}(f(\bar{X}^{(h_k)}), f(\bar{X}^{(h_{k+1})}))$ will be high.

Steps:

i) Use MC simulation to compute $\mathbb{E}(f(\bar{X}^{(h_1)}))$

ii) Use variance reduction to compute $\mathbb{E}(f(\bar{X}^{(h_2)}))$ based on $f(\bar{X}^{(h_1)})$.

iii) Repeat it on $(f(\bar{X}^{(h_\ell)}), f(\bar{X}^{(h_{\ell+1})}))$.

Rmk: We can use Brownian bridge since the BM on finer grid can base on the coarse one.

Fix $N \in \mathbb{N}$. $N > 1$. $h_\ell := N^{-\ell} T$. $\ell = 0, \cdots L$. Let $P_\ell := f(\bar{X}^{(h_\ell)})$. $I_\ell := M_\ell^{-1} \sum_{i}^{M_\ell} (P_\ell^{(i)} - P_{\ell-1}^{(i)})$ and assume I_ℓ's are indept.

Thm. If $\exists$ const's $\alpha \geq \frac{1}{2}$. $C_1, C_2, \beta > 0$ satisfy
$\mathbb{E}(f(X_T) - P_\ell) \leq C_1 h_\ell^{\alpha}$. $Var(I_\ell) \leq C_2 h_\ell^{\beta} / M_\ell$

Then: $\exists L \in \mathbb{N}$ and $(M_i)_0^L$. s.t. the multilevel estimate $I = \sum_0^L I_\ell$ satisfy:

$(\mathbb{E}\{(I - \mathbb{E}(f(X_T)))^2\})^{\frac{1}{2}} \leq \varepsilon$.

And computational cost C satisfies:

$$C \leq \begin{cases} C_3 \varepsilon^{-2} & . \quad \beta > 1 \\ C_3 \varepsilon^{-2} (\log \varepsilon)^2 & . \quad \beta = 1 \\ C_3 \varepsilon^{-2-(1-\beta)/\alpha} & . \quad 0 < \beta < 1 \end{cases}$$

Rmk: $\mathbb{E}(I) = \sum \mathbb{E}(I_\ell)$

$= \sum \mathbb{E}(P_\ell) - \mathbb{E}(P_{\ell-1}) = \mathbb{E}(P_L)$

which explains the error comes from the finest term

<u>Cor.</u> Given Euler method of weak order one and strong order $1/2$. Choose $L = \log \varepsilon^{-1} / \log N + O(1)$ $(\varepsilon \to 0)$ and $M_\ell \propto \varepsilon^{-2}(L+1) \cdot h_\ell$. Then: for error $O(\varepsilon)$ of I, its comp. cost $= O(\varepsilon^{-2}(\log \varepsilon)^2)$

<u>Rmk:</u> Choose $\alpha = \beta = 1$ Thm above. We can obtain this cor.

<u>Pf:</u> 1) ~~Set~~ $L := \left\lceil \dfrac{\log(\sqrt{2}\, C_1 T \varepsilon^{-1})}{\log N} \right\rceil$

$\Rightarrow C_1 h_L \in (\varepsilon/\sqrt{2}m, \varepsilon/\sqrt{2}]$.

$\int_0 : (\mathbb{E}(I) - \mathbb{E}(f(X_T)))^2 \lesssim_L \varepsilon^2/2$.

Choose $M_\ell := \lceil 2\varepsilon^{-2}(L+1) C_2 h_\ell \rceil$

$Var(I) = \Sigma\, Var(I_\ell)$

$\leq C_2 \Sigma\, h_\ell / m_\ell \lesssim \varepsilon^2/2$.

$\int_0 : MSE(I) = Var(I) + (\mathbb{E}(I) - \mathbb{E}(f(X_T)))^2 \lesssim \varepsilon^2$

2) $L + 1 \lesssim C \log \varepsilon^{-1}$. $M_\ell \leq 2\varepsilon^{-2}(L+1) C_2 h_\ell + 1$

$\Rightarrow \text{Cost} \le C \sum_0^L M_\ell / h_\ell \le \sum_{\ell=0}^{L} (2\varepsilon^{-2}(L+1)C_2 + h_\ell^{-1})$

$\le 2\varepsilon^{-2}(L+1)^2 C_2 + \sum h_\ell^{-1}$

$\sum h_\ell^{-1} = h_L^{-1} \sum N^{-\ell} = h_L^{-1} \dfrac{N^{-(L+1)} - 1}{N^{-1} - 1}$

$< h_L^{-1} N/N-1 \le N^2 \sqrt{2}\, C_1 \varepsilon^{-1} / (N-1)$

So: $\text{Cost} \le C \varepsilon^{-2} (\log \varepsilon^{-1})^2$.

<u>Rmk</u>: Set $L = \lceil \log(\sqrt{2} C_1 T^\alpha \varepsilon^{-1}) \rceil / \alpha$

$M_\ell = \lceil 2\varepsilon^{-2}(L+1) C_2 h_\ell^\beta \rceil$

We can prove general (α, β).

② <u>Stochastic Taylor schemes:</u>

Note in deterministic case, high order Taylor expansion can lead to high order scheme while in stochastic case, due to BM behave differently with BV, we can try to obtain it by Itô's.

Assume: $dX_t = \sum_0^d V_i(X_t) \circ dB_t^i$. (Stratonovich)

By Itô's: $f(X_t) = f(X_0) + \sum_0^d \int_0^t V_i f(X_s) \circ dB_s^i$

Apply it on $x \mapsto V_i f(x)$, we have:

$V_i f(X_s) = V_i f(X_0) + \sum_{j=0}^{d} \int_0^s U_j V_i f(X_u) \circ dB_u^j$. $\forall i$.

insert back on $f(X_t) = f(X_0) + \square$:

$$f(X_t) = f(X_0) + \sum_0^d V_i f(X_0) \int_0^t \circ dB_t^i$$
$$+ \sum_{i,j}^{d} \int_0^t \int_0^s V_j V_i f(X_u) \circ dB_u^j \circ dB_s^i$$

$\Rightarrow$ We can get high order expansion by iteration. For order of expansion:

Note $B_t^0 = t \sim t$. $B_t^i \sim \sqrt{t}\, B_1^i$. $\forall i > 0$.

$\int_0 B_t^I \overset{\Delta}{=} \int_{0 \le t_1 \cdots \le t_k \le t} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k} \sim t^{\deg(I)/2} B_1^I$.

where $I = (i_1, \cdots i_k) \in \{0, \cdots d\}^k$ and $\deg(I) =$ $k + \#\{1 \le j \le k \mid i_j = 0\}$.

<u>Lm.</u> If f. $(V_i) \in C_B^{m+1}$. $X_0 = x \in \mathbb{R}^n$. Then:

$$f(X_t) = f(x) + \sum_{\substack{I \in \{0, \cdots d\}^k, k \le m \\ \deg(I) \le m}} V_{i_1} \cdots V_{i_k} f(x) B_t^I + R_m^{t,f,x}$$

St. $\forall t < 1$,

$$\sup_x \mathbb{E}(|R_m^{t,f,x}|^2)^{\frac{1}{2}} \le C t^{\frac{m+1}{2}} \sup_{m \le \deg(I) \le m+2} \|V_{i_1} \cdots V_{i_k} f\|_\infty$$

<u>Pf</u>: Expand $f(X_t)$ up to order m and transf. back to Itô's integral to get the estimate.

Rmk: For Itô case, $dX_t = \sum_0^d V_i(X_t) dB_t^i$.

Let $\tilde{V}_0 f(x) = Vf(x) + \frac{1}{2} \sum_1^d V_i(x)^T Hf(x) V_i(x)$

where Hf is Hessian matrix of f.

Rmk: $\tilde{V}_0$ isn't vector field! (2^{nd}-order)

$\Rightarrow f(X_t) = f(X_0) + \sum_0^d \int_0^t \tilde{V}_i f(X_s) dB_s^i$ where

$\tilde{V}_i = V_i$. $i = 1, 2, \cdots d$.

Def: i) $A_m := \{ I = (i_1, \cdots i_k) \in \{0, \cdots d\}^k \mid 1 \le k \le m, \deg(I) \le m \}$. $A_m^* = A_m \cup \{ k = \#\{ j \mid i_j = 0 \} = \frac{m+1}{2} \}$.

Rmk: A_m^* contains the term with

order $= \frac{m+1}{2} \cdot 2 = m+1$.

ii) $V_I = V_{i_1} \cdots V_{i_k}$ for $I = (i_1, \cdots i_k) \in \{0, \cdots d\}^k$

Rmk: By Thm above:

$$f(X_t) = f(x) + \sum_{I \in A_m} V_I f(x) B_t^I + R_m^{t,f,x}$$

$$= f(x) + \sum_{I \in A_m} \tilde{V}_I f(x) \tilde{B}_t^I + \tilde{R}_m^{t,f,x}$$

(where $\tilde{B}_t^I = \int_\square dB_{t_1}^{i_1} \cdots dB_{t_k}^{i_k}$)

Def: Strong stochastic Taylor scheme of order m for Itô-SDE is: $\bar{X}_0 = X_0$. and

$\tilde{X}_{j+1} = \tilde{X}_j + \sum_{I \in A_m^*} \tilde{V}^I(x) \Delta \tilde{B}_j^I$ where $\Delta \tilde{B}_j^I :=$
$\int_{t_j \le s_1 \le \cdots \le s_k \le t_{j+1}} dB_{s_1}^{i_1} \cdots dB_{s_k}^{i_k}$. $I = (i_1 \cdots i_k)$.

<u>Thm.</u> If $\forall I \in A_m$, $\tilde{V}^I \in C_b^2$. Then the strong stochastic Taylor scheme $\to X_T$ sol. of Itô-SDE with strong order $m/2$.

<u>Rmk:</u> Euler scheme is strong Taylor scheme with $m = 1$.

<u>Def:</u> Weak stochastic Taylor scheme of order m for Itô-SDE is: $\tilde{X}_0 = X_0$. and
$\tilde{X}_{j+1} = \tilde{X}_j + \sum_{I \in \bigcup_{1 \le k \le m} \{0, \cdots, d\}^k} \tilde{V}^I(x) \Delta \tilde{B}_j^I$

<u>Thm.</u> $(V_i) \in C^{2(m+1)}$. Then for $\mathcal{G} = \{f \in C^{2(m+1)}$, its deri. is poly. bdd$\}$. the weak Taylor scheme have weak order m.

<u>Rmk:</u> i) Euler scheme is weak Taylor scheme with $m = 1$ as well.

ii) We can also define the weak Stratonovich Taylor scheme :

$$\bar{X}_{j+1} = \bar{X}_j + \sum_{I \in A_m} V^I(\bar{X}_j) \Delta B_j^I$$

it has weak order $(m-1)/2$ for g if $f, (V_i)$ regular enough.

<u>e.g.</u> (Milstein scheme)

It's a scheme with weak order one and strong order two, given by:

$$\overline{X}_{j+1} = \overline{X}_j + V(\overline{X}_j)\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_j)\Delta B_j^i + \sum_{(i_1,i_2)\in\{1,\dots,d\}^2} V^{(i_1,i_2)}(\overline{X}_j)\Delta\widetilde{B}_j^{(i_1,i_2)}.$$

Here, $V^{(i_1,i_2)}(x) = DV_{i_2}(x) \cdot V_{i_1}(x)$ and

$$\Delta\widetilde{B}_j^{(i_1,i_2)} = \int_{t_j}^{t_{j+1}} B_s^{i_1} dB_s^{i_2}.$$

<u>Rmk</u>: i) For $i_1 = i_2 = i$. $\Rightarrow \Delta \widetilde{B}_j^{(i,i)} = (\Delta B_j^i)^2 - \Delta t_j$

For $i_1 \neq i_2$. there is no explicit formula for the increments.

ii) For $[V_{i_1}, V_{i_2}] = 0$. Note:

$$B_t^{i_1} B_t^{i_2} = \int B_s^{i_1} dB_s^{i_2} + \int B_s^{i_2} dB_s^{i_1} + [B^{i_1}, B^{i_2}]_t$$

$\Rightarrow$ We can just calculate:

$$B_t^{i_1} B_t^{i_2} - [B^{i_1}, B^{i_2}]_t.$$

Next, we consider the sampling of B^I:

e.g. (Milstein scheme in 2-dim)

We need to sample (B_t^1, B_t^2, A_t).

A_t is Lévy's area of B^1, B^2

Rmk: There exists explicit formula for ch.f. of (B^1, B^2, A). But no way to obtain its density. So Accept-Reject method do not work.

We can use def of Ito's integral in Riemann's sum to approxi.:

$B_t^i \approx \sum_1^N \Delta B_k^i$, $i = 1, 2$, $\Delta = (t/N)$,

$A_t \approx \sum_1^N B_{t_{j-1}}^1 \Delta B_j^2 - B_{t_{j-1}}^2 \Delta B_j^1$

Rmk: It's not a competitive algo. and we can consider moment matching (up to two)