Fourier Method.

To compute $I(f, X) := \mathbb{E}(f(X))$:

$$\mathbb{E}(f(X)) = \int f \, d\mathbb{P}_X \to \int T^{-1}(T(f)) \, d\mathbb{P}_X$$

$$\to \int T(f)(u) \, T(\mathbb{P}_X)(du)$$

where T is suitable transf.

<u>Rmk</u>: This method can be used when RHS is much simpler than LHS i.e. when $T(f)$. $T(\mathbb{P}_X)$ is well-known explicitly. <u>e.g.</u> Lévy's process.

(1) <u>Preliminary</u>:

<u>Def</u>: $\hat{f}(x) := \int_{\mathbb{R}^1} e^{iux} f(u) \, du$. $x \in \mathbb{C}$.

<u>Rmk</u>: Recall $f \in L^1 \Rightarrow \hat{f} \in C_0$

$$(\hat{f}(u) = \frac{1}{2} \int_{\mathbb{R}^1} e^{iux} (f(x) - f(x - \frac{\pi}{u})) dx)$$

<u>Denote</u>: $L^1_{BC}(\mathbb{R}^1) = C_B(\mathbb{R}^1) \cap L^1(\mathbb{R}^1)$.

<u>Lem</u>. For $f \in L^1$. $h \in L^1$

i) $g(x) = f(x) e^{iax} \Rightarrow \hat{g}(u) = \hat{f}(u+a)$

$g(x) = f(x-a) \Rightarrow \hat{g}(u) = e^{iua}\hat{f}(u)$

ii) $g(x) = f(x/\lambda) \Rightarrow \hat{g}(u) = \lambda \hat{f}(\lambda u)$

iii) $g(x) = \overline{f(-x)} \Rightarrow \hat{g}(u) = \overline{\hat{f}(u)}$

iv) $\widehat{f * h} = \hat{f} \cdot \hat{h}$

v) $g(x) = ix f(x) \in L^1 \Rightarrow \hat{f} \in C^1, (\hat{f})' = \hat{g}$

vi) $f, f' \in L^1_{loc}(\mathbb{R}^1) \Rightarrow \hat{f'}(u) = -iu\hat{f}(u)$.

vii) $f \in C^2(\mathbb{R}^1)$ and $f, f', f'' \in L^1_{loc}(\mathbb{R}^1)$

$\Rightarrow \hat{f} \in L^1_{loc}(\mathbb{R}^1)$.

Pf: Only prove vii):

Note $\hat{f''}(u) = -u^2\hat{f}(u) \in C_0$ by vi).

$\Rightarrow (1+u^2)\hat{f}(u)$ is bdd.

So: $\int \hat{f} \le C \int 1/_{1+u^2} < \infty$.

Thm. (inversion formula)

$\check{g}(u) \stackrel{\Delta}{=} \frac{1}{2\pi}\int_{\mathbb{R}^1} e^{-iux} g(x)dx$ for $g \in L^1$.

Then: if $f \in L^1_{loc}, \hat{f} \in L^1 \Rightarrow \forall x \in \mathbb{R}^1$.

$f(x) = (\hat{f})^{\vee}(x)$

(2) Optional pricing:

Def: For $f_k(x) = e^{-kx} f(x)$. $M_X(u) = \mathbb{E}(e^{uX})$.

$\mathcal{I} := \{k \in \mathbb{R}^1 : f_k \in L^1_{loc},\ \hat{f}_k \in L^1\}$.

$\mathcal{J} := \{k \in \mathbb{R}^1 : M_X(k) < \infty\}$.

Thm. $\mathcal{R} := \mathcal{I} \cap \mathcal{J} \neq \emptyset$. Let $k \in \mathcal{R}$. Then:

$$I(f, X) = \frac{1}{2\pi} \int_{\mathbb{R}^1} M_X(k - iu) \hat{f}(u + ik)\, du.$$

Pf: Note $\hat{f}_k \in L^1$. by inversion:

$$f_k(x) = \frac{1}{2\pi} \int_{\mathbb{R}^1} e^{-ixu} \hat{f}_k(u)\, du.$$

So: $LHS = \int e^{kx} f_k(x)\, dP_X$

$$= \int e^{kx} \cdot \frac{1}{2\pi} \int e^{-ixu} \hat{f}_k\, dP_X.$$

$$= \frac{1}{2\pi} \int M_X(k - iu) \hat{f}(u + ik)\, du$$

follows from Fubini Thm.

$$\left(\int e^{kx} |e^{iux} \hat{f}_k|\, du\, dP_X \leq \int e^{kx} \|\hat{f}\|_{L^1}\, dP_X \right.$$

$$\left. \leq M_X(k) \|\hat{f}\|_{L^1} \right)$$

Rmk: i) Actually, $\mathcal{R} \neq \emptyset \Rightarrow f \in C(\mathbb{R}^1)$.

For discontin. case:

Set $\mathcal{I}' := \{k \in \mathbb{R}^1 : f_k \in L^1(\mathbb{R}^1)\}$.

$\mathcal{J}' := \{k \in \mathbb{R}^1 : M_X(k) < \infty,\ M_X(k - i\cdot) \in L^1\}$

Then we can replace "$R \in \mathcal{R} \neq \emptyset$" by "$R \in \mathcal{R}' := \mathcal{J}' \cap \mathcal{I}' \neq \emptyset$" (But it implies $e^{Ru} P_X(du)$ admits a conti. bdd Lebesgue density e. It's a trading on conti. of f and dist. of X / integrability of $\hat{f}$ and M_X)

ii) Set $\mathcal{I}_{min} := \{R \in \mathbb{R}' : f_R \in L^1(\mathbb{R}')\}$.

$\mathcal{J}_{min} := \{R \in \mathbb{R}' : M_X(R) < \infty\}$.

The minimal assumption of thm above is : $R \in \mathcal{I}_{min} \cap \mathcal{J}_{min} \neq \emptyset$.

Then the formula exists as a pointwise limit.

iii) The only hard part to check in the condition is $f_R \in L^1_{loc}$. As for "$\hat{f}_R \in L^1(\mathbb{R}')$": We have:

<u>Lem.</u> $g \in H^1(\mathbb{R}') \Rightarrow \widehat{g'}(w) = -iw\hat{g}(w)$. $\hat{g}, \widehat{g'} \in L^2$.

<u>Cor.</u> $g \in H^1(\mathbb{R}') \Rightarrow \hat{g} \in L^1(\mathbb{R}')$.

$\underline{Pf}$: $\int |\hat{g}|^2(1+|u|^2) = \int |\hat{g}|^2 + |\hat{g'}|^2 < \infty$

$$S_0 := \int |\hat{g}| \leq \left(\int |\hat{g}|^2 (1+|u|)^2\right)^{\frac{1}{2}} \cdot \left(\int 1/(1+|u|)^2\right)^{\frac{1}{2}} < \infty.$$

Applications:

① Consider (S_t) price of assets modeled as exponential semimart. $S_t = S_0 e^{X_t}$.

Next, assume $r=0$. S_t is $\mathbb{P}$-mart.

$\Rightarrow F(S_T) = f(X_T + \log S_0)$ payoff F of S_T where $f = F \circ \exp$. We want to compute:

$$\mathbb{E}(F(S_T)) = \mathbb{E}(f(X_T + \log S_0))$$
$$= \frac{1}{2\pi} \int M_{X_T + \log S_0}(R-iu) \hat{f}(u+iR)\, du$$
$$= \frac{1}{2\pi} \int S_0^{R-iu} M_{X_T}(R-iu) \hat{f}(u+iR)\, du.$$

Next, we consider vanilla option, i.e. it can't be exercised earlier before expire time T and only depends on S_T.

$\underline{e.g.}$ I) (Call option)

$f(x) = (e^x - k)_+$ is payoff func.

$\Rightarrow \hat{f}(z) = K^{1+iz} / iz(1+iz)$.

Note $\{R \in (1, \infty)\} \subset \{f_R \in L^1_{loc} \cap L^2\}$

And $f_R'(x) = e^{-Rx}(e^x - Re^x + RK) I_{\{x > \ln K\}}$

in weak sense. $\Rightarrow f_R' \in L^2$ as well.

So: $\hat{f_R} \in L^1$ since $f_R \in H^1(\mathbb{R}')$.

$\Rightarrow (1, \infty) \subset I$.

<u>Rmk</u>: For put option $f(x) = (k - e^x)_+$.

We have $(-\infty, 0) \subset I$.

ii) Fourier method is quite suitable for exp. Lévy process. Since we know M_X explicitly.

Assume its triplet (b, c, ν) and $M_X < \infty$ on $\{w \in \mathbb{C} : Re(w) \in [a, b]\}$.

i.e. $J = [a, b]$. So if consider f is call option above and $[a, b] \cap (1, \infty) \neq \emptyset$. Then we can use the formula.

Rmk: $[a,b] \supset [0,1]$ in fact.

② **Computation of Greek:**

We want to find delta Δ for ①:

$\Delta f(X, S_0) = \partial/\partial S_0 \, \mathbb{E}(f(X_T + \log S_0))$

Thm: If "i) $|u||M_{X_T}(R-iu)| \in L^1$ and $\hat{f}(\cdot + iR)$ is bdd" or "ii) $|u||\hat{f}(u+iR)| \in L^1$ and $M_{X_T}(R-i\cdot)$ is bdd" holds. Then:

$$\Delta f(X, S_0) = \frac{1}{2\pi}\int_{\mathbb{R}^1} S_0^{R-1-iu} M_{X_T}(R-iu) \frac{\hat{f}(u+iR)}{(R-iu)^{-1}} du$$

Pf: Apply DCT. it follows from:

$$|\partial/\partial S_0 \square| \le C(1+|u|)|M_{X_T}(R-iu)||\hat{f}(u+iR)|$$

Rmk: Cond i) implies $\mathbb{P}_{X_T}$ admits a density $\in C^1$.

③ **Multidimension:**

For $f: \mathbb{R}^d \to \mathbb{R}^1$ payoff and (X_t) d-dim r.v.'s. Replace "$R \cdot X$" by "$\langle R, X\rangle$". $R \in \mathbb{R}^d$

Set $\mathcal{R}, \mathcal{R}' \subset \mathbb{R}^d$ as above.

Thm. If $\mathcal{R}\neq\emptyset$ or $\mathcal{R}'\neq\emptyset$. $R\in\mathcal{R}'\cup\mathcal{R}$. Then:

$$I(f,X)=(2\pi)^{-d}\int_{\mathbb{R}^d} M_X(R-iu)\,\hat{f}(u+iR)\,du$$

Cor. $\mathbb{E}(F(S_T))$

$$=(2\pi)^{-d}\int_{\mathbb{R}^d} e^{\langle R-iu,\log S_0\rangle} M_{X_T}(R-iu)\,\hat{f}(u+iR)\,du$$

Rmk: It'll suffer curse of dim. So we mostly consider $d\leq 3$.

Lem. $f_i:\mathbb{R}'\to\mathbb{R}'$. Set $f(x)=\prod_{i=1}^{d} f_i(x_i)$. Then:

$\mathcal{I}'\supset\bigotimes_1^d \mathcal{I}_i'$.

e.g. C (call option)

$f(x)=(e^{x_1}\wedge\cdots\wedge e^{x_d}-K)^+$

$\Rightarrow\hat{f}(z)=-K^{1+i\sum_1^d z_k}\Big/(-1)^d(1+i\sum_1^d z_k)\prod_1^d(iz_k)$

$\mathcal{I}'=\{R_i>0,\ \forall i\leq d.\ \sum_1^d R_i>1\}$.

Rmk: For put option $(K-e^{x_1}\wedge\cdots\wedge e^{x_d})^+$

$\Rightarrow\hat{f}(z)=K^{1+i\sum_1^d z_k}\Big/(1+i\sum_1^d z_k)\prod_1^d(iz_k)$

and $\mathcal{I}'=\{R_i<0,\ \forall i\leq d\}$.

(3) Fast Fourier Transf.:

Set $s=\log S_0$. Next, we want to implement the formula into computer:

$$O_f(s) = \mathbb{E}(F(S_T)) = \frac{e^{ks}}{2\pi}\int_{\mathbb{R}} e^{-ius} \mu_{X_T}(k-iu) \hat{f}(u+ik)\, du.$$

$$\triangleq e^{ks}/2\pi \cdot \int_{\mathbb{R}} e^{-ius} \psi(u) du.$$

$$\approx \frac{e^{ks}}{2\pi}\int_a^b e^{-ius}\psi(u)\, du.$$

(i.e. fix on finite interval to approxi.)

Set $\eta = \frac{b-a}{N-1}$. $u_k = a+\eta k$. $0 \le k \le N-1$.

$$\int_a^b e^{-ius}\psi(u) \approx \eta\Big(\frac{e^{-ias}\psi(a)}{2} + \sum_1^{N-2} e^{-iu_k s}\psi(u_k) + e^{-ibs}\psi(b)/2\Big) = \sum_0^{N-1}{}'' e^{-iu_k s}\psi(u_k)$$

We also choose a uniform grid on s-domain $[-\frac{\lambda N}{2}, \frac{\lambda N}{2} - \lambda]$: let $\beta = -\frac{\lambda N}{2}$.

$s_j = \beta + \lambda j$. $0 \le j \le N-1$. So: we want to compute: $\sum_{k=0}^{N-1} e^{-i(a+k\eta)\lambda j} e^{i\beta u_k}\psi(u_k)\eta$. $j \le N-1$

We choose λ, η. s.t. $\lambda\eta = 2\pi/N$. (Nyquist rel.)

let $\phi_j := \sum_{k=0}^{N-1} e^{-i\frac{2\pi}{N}kj} \varphi_k$. $\varphi_k = e^{i\beta u_k}\psi(u_k)$.

So: $O_f(s_j) \approx e^{\beta k - \lambda j(k-ia)} \eta \phi_j / 2\pi$.

<u>Rmk</u>: $\phi = (\phi_0 \cdots \phi_{N-1})$ is discrete Fourier

transf. of vector $\varphi = (\varphi_0, \cdots \varphi_{N-1})$

We want to find a fast algo. to compute $\hat{\varphi}$.

Next, we introduce fast Fourier transf. (FFT). It can reduce the computation cost $\sim N^2$ above to cost $\sim N \log_2 N$.

Let $\omega_N := e^{-2\pi i/N}$ and define the $N \times N$-matrix T_N by

$$T_N := \begin{pmatrix} \omega_N^0 & \omega_N^0 & \omega_N^0 & \cdots & \omega_N^0 \\ \omega_N^0 & \omega_N^1 & \omega_N^2 & \cdots & \omega_N^{N-1} \\ \omega_N^0 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_N^0 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{pmatrix}. \Rightarrow \hat{\varphi} = T_N \varphi$$

<u>Lem.</u> $\varphi \in \mathbb{C}^{2N}$. $\hat{\varphi} = T_{2N} \varphi$. let $\varphi' := (\varphi_1, \varphi_3, \cdots \varphi_{2N-1})$, $\varphi'' := (\varphi_2, \varphi_4, \cdots \varphi_{2N})$, $\hat{\varphi}' := (\hat{\varphi}_1, \hat{\varphi}_2, \cdots \hat{\varphi}_N)$, $\hat{\varphi}'' := (\hat{\varphi}_{N+1}, \cdots \hat{\varphi}_{2N})$. And $D_N := \operatorname{diag}\{W_{2N}^0, \cdots W_{2N}^{N-1}\}$. Then:

$\hat{\varphi}' = c + d$, $\hat{\varphi}'' = c - d$ where $c = T_N \varphi'$ and $d = D_N T_N \varphi''$.

<u>Pf:</u> Using $W_N^{jk} = W_{2N}^{2jk}$

It forms a divide-and-conquer algo.:

Algorithm 7.26 (FFT). *Assume that $N = 2^J$, $J \geqslant 1$. Given $\phi \in \mathbf{C}^N$, apply the following recursive algorithm to compute its discrete Fourier transform $\Phi = T_N\phi$:*

1. *If $N = 2$ go to 2, otherwise: split ϕ into ϕ' and ϕ'' as in Lemma 7.25, apply the FFT to compute $c = T_{N/2}\phi'$, $d = D_{N/2}T_{N/2}\phi''$ and return $\Phi = (\Phi', \Phi'')$ given by $\Phi' = c + d$ and $\Phi'' = c - d$.*
2. *If $N = 2$ compute $\Phi = T_2\phi$ directly.*

Let $N_k = 2^k$. $1 \le k \le J$. implement J times.

Lem. $N = 2^k$. C is cost of floating point operations (addition, multipli. ...). Then: Computation work $W(N) \le C(\frac{3}{2}\log_2 N + \frac{1}{2})N$.

Pf: Note for FFT in N-dim. we need one vector addition and one subtraction in $N/2$-dim. With one elementwise multiplicate of two vectors in dim $N/2$.

$\Rightarrow W(N) \le 2W(N/2) + \frac{3}{2}CN.$

$W(2) \le 4C.$

let $\tilde{W}(N) = W(N)/CN.$

So: $\tilde{W}(N) \le \tilde{W}(N/2) + \frac{3}{2}$

$\le (k-1)\frac{3}{2} + \tilde{W}(2)$

$\le \frac{3}{2}k + \frac{1}{2}$

where $k = \log_2 N$.

Rmk: i) We can also compute the inverse discrete Fourier transf.

ii) Variant of FFT exists. i.e. N doesn't need to be 2^k.

(4) Cosine-series expansion:

Note for even func f. We have:

$\hat{f}(z) = 2\int_0^\infty f(x)\cos(xz)\,dx$.

Assume q_T density of X_T decay very fast to 0. So: WLOG. $\mathrm{Supp}(q_T) \subset [0, \pi]$.

Rmk: Recall for locally cpt abelian group. G, the dual group $\hat{G}$ consists of all characters of G. i.e. all conti. group homo. from G to $\mathbb{T} \subset \mathbb{C}'$.

a) $G = \mathbb{R}' \Rightarrow \hat{G} \cong \mathbb{R}'$. i.e. $\chi(x) = e^{iux}$. $u \in \mathbb{R}'$.

b) $G = [-\pi, \pi] \Rightarrow \hat{G} \cong \mathbb{Z}$, i.e. $\chi(x) = e^{inx}$. $n \in \mathbb{Z}$.

Let μ denote Haar measure on G.

$\hat{f}(\chi) := \int_G f(x)\overline{\chi(x)}\mu(dx) \in C_B(\hat{G})$ is

Fourier transf. of $f \in L^1(G;\mathbb{C})$.

a) $G = \mathbb{R}^1 \Rightarrow \mu = \mathcal{L}$. So $\hat{f} = \int_{\mathbb{R}^1} e^{-iux} f(x)$

b) $G = [-\pi, \pi] \Rightarrow \mu = \mathcal{L}|_{[-\pi,\pi]} / 2\pi$. So c_n

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$. $\hat{f} = \sum c_n e^{-inu}$.

Consider cosine represent of $q = q_T$:

$q(\theta) = \sum_1^{\infty} A_k \cos(k\theta) + \frac{1}{2} A_0$. $A_k = \frac{2}{\pi} \int_0^{\pi} q(\theta) \cos k\theta d\theta$

Rmk: For q supp. on $[a,b]$. By change vari:

$q(\theta) = \sum_1^{\infty} A_k \cos(k\pi \frac{\theta - a}{b-a}) + \frac{1}{2} A_0$ with

$A_k = \frac{2}{b-a} \int_a^b q(x) \cos(k\pi \frac{x-a}{b-a}) dx$.

If we know $\phi = \hat{q}$ but not q. First

Let $\int_a^b e^{iux} q(x) dx = \phi_1 \approx \phi$.

$\Rightarrow A_k = \frac{2}{b-a} \mathrm{Re}(\phi_1 (\frac{k\pi}{b-a}) e^{-ika\pi/(b-a)})$

$\approx \frac{2}{b-a} \mathrm{Re}(\phi (\frac{k\pi}{b-a}) e^{-ika\pi/(b-a)}) =: F_k$

So: $q(x) \approx q_1(x) \stackrel{\Delta}{=} \sum_1^{N-1} F_k \cos(k\pi \frac{x-a}{b-a}) + \frac{1}{2} F_0$.

Rmk: There're three different errors:

a) Truncated the integral on $[a,b]$.

b) Replace A_k by F_k.

c) Truncate infinite sum by finite sum $\sum_0^{N-1}{}'$.

So next we can consider the option valuation of f: $C(S_0, T) = e^{-rT}\int_{-\infty}^{+\infty} f(x) q_T(x)$

Replace q_T by approxi. above. We have:

$$C(S_0, T) \approx e^{-rT} \sum_0^{N-1}{}' \mathrm{Re}\left(\phi_T\left(\frac{k\pi}{b-a}\right) e^{-ik\pi\frac{a}{b-a}}\right) C_k.$$

where $\phi_T = \hat{q}_T$, $C_k = \frac{2}{b-a}\int_a^b f(x)\cos\left(k\pi\frac{x-a}{b-a}\right)dx$

Rmk: i) The error of approxi. mostly depend on smoothness of density q_T:

a) q_T smooth on $[a, b]$. $\Rightarrow$ ε decay exponentially $\sim e^{-\nu(N-1)}$

b) $q_T \in C'$ on $[a, b] \Rightarrow \varepsilon$ will decay algebraically $\sim (N-1)^{-\beta}$

ii) Choice of a, b will depend on cumulants c_n of dist.:

(7.41) $$a = c_1 - L\sqrt{c_2 + \sqrt{c_4}}, \quad b = c_1 + L\sqrt{c_2 + \sqrt{c_4}}$$

with $L = 10$.

ii) For use the approx.: We need to compute C_k at first. In some case C_k is explicit.

e.g. (Call option)

$f(x) = (K(e^x - 1))_+$. $x = \log(S_T/K)$

log-moneyness. Then:

$$C_k^{\text{call}} = \frac{2}{b-a} K(\chi_k(0,b) - \psi_k(0,b)),$$

with

$$\chi_k(c,d) := \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{d-a}{b-a}\right)e^d - \cos\left(k\pi\frac{c-a}{b-a}\right)e^c + \right.$$
$$\left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right]$$

and

$$\psi_k(c,d) := \begin{cases} \left(\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right)\frac{b-a}{k\pi}, & k \neq 0, \\ d-c, & k = 0. \end{cases}$$

For the put-option, we obtain

$$C_k^{\text{put}} = \frac{2}{b-a} K(\psi_k(a,0) - \chi_k(a,0)).$$

Rmk: It's only valid for options written in log-moneyness.