Numerics for PDEs

(1) <u>BS-PDE:</u>

Consider BS model: $dS_t/S_t = r\,dt + \sigma\,dB_t$. $S_0 = s$.

$\Rightarrow$ price $u(t,x) = \mathbb{E}(e^{-r(T-t)} f(S_T) \mid S_t = x)$ of payoff f on European option will satisfy BS-PDE (also parabolic PDE):

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(t,x) + rx\frac{\partial}{\partial x}u(t,x) = ru(t,x) \\ u(T,x) = f(x). \end{cases}$$

For American option, price $\tilde{u}$ satisfies:

(6.2a) $$\frac{\partial}{\partial t}\tilde{u}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\tilde{u}(t,x) + rx\frac{\partial}{\partial x}\tilde{u}(t,x) - r\tilde{u}(t,x) \leqslant 0,$$

(6.2b) $$\tilde{u}(t,x) \geqslant (K-x)_+,$$

(6.2c) $$\tilde{u}(T,x) = (K-x)_+,$$

<u>Rmk:</u> We have equality in (6.2a) if inequi. in (6.2b) is strict. It's a free boundary problem. i.e. $\exists X_0(t)$. s.t. $\tilde{u}$ solves (6.2a) with equality on $(X_0(t), \infty)$ (behaves like European option). And $\tilde{u}(t,x) = (K - X_0(t))_+$ else.

To solve the two problems numerically. We first need to simplify the PDEs:

Let $y = \log(x/K)$. $\tau = \frac{1}{2}\sigma^2(T-t)$. $q = 2r/\sigma^2$

$v(\tau, y) = \frac{1}{K}\exp(\frac{1}{2}(q-1)y + (\frac{1}{4}(q-1)^2 + q)\tau) u(t, x)$

and denote $\tilde{v}$ for $\tilde{u}$ in same way.

Rmk: $v(\tau, y) \to \exp(\frac{1}{2}(q-1)y + \frac{1}{4}(q-1)^2\tau)$ $(y \to -\infty)$

$v(\tau, y) \to 0$ $(y \to +\infty)$.

v satisfies: $\frac{\partial}{\partial \tau} v = \frac{\partial^2}{\partial y^2} v$.

$v(0, y) = (e^{\frac{1}{2}(q-1)y} - e^{\frac{1}{2}(q+1)y})_+$.

For American option:

let $g(y,\tau) := \exp\left(\frac{1}{4}(q+1)^2\tau\right)\left(e^{\frac{1}{2}(q-1)y} - e^{\frac{1}{2}(q+1)y}\right)_+$, then

(6.6a) $$\left(\frac{\partial}{\partial \tau}\tilde{v}(\tau,y) - \frac{\partial^2}{\partial y^2}\tilde{v}(\tau,y)\right)(\tilde{v}(\tau,y) - g(\tau,y)) = 0, \quad \frac{\partial}{\partial \tau}\tilde{v}(\tau,y) - \frac{\partial^2}{\partial y^2}\tilde{v}(\tau,y) \geqslant 0,$$

(6.6b) $$\tilde{v}(\tau,y) \geqslant g(\tau,y), \quad \tilde{v}(0,y) = g(0,y),$$

(6.6c) $$\tilde{v}(\tau,y) = g(\tau,y) \text{ for } y \to -\infty, \quad \tilde{v}(\tau,y) = 0 \text{ for } y \to \infty.$$

(2) Finite difference method:

Consider $u(t, x)$ on $[0, T] \times [a, b]$. let

$\Delta t = T/N$. $\Delta x = (b-a)/m$. $t_i = i\Delta t$. $x_i = a + i\Delta x$.

$v(t_i, x_j) := v_{i,j}$. $0 \le i \le N$. $0 \le j \le m$.

Rmk: i) The limit behavior of V is necessary to set boundary value condition of V at $x = a, b$.

(e.g. let $V(t,a) = V(t,-\infty)$, $V(t,b) = V(t,+\infty)$, for $-a, b$ large)

ii) It still suffers curse of dim. since in $\mathbb{R}^n$. We need to set m^n nodes with same mesh Δx.

① Explicit case:

We use D_t^+, D_x^2 replace $\partial/\partial t$, $\partial^2/\partial x^2$:

Denote $\lambda := \Delta t / (\Delta x)^2$. We have:

$$v_{0,j} = \left(e^{\frac{1}{2}(q-1)x_j} - e^{\frac{1}{2}(q+1)x_j}\right)_+, \quad j = 0, \ldots, M, \tag{6.9a}$$

$$v_{i+1,j} = v_{i,j} + \lambda(v_{i,j+1} - 2v_{i,j} + v_{i,j-1}), \quad i = 0, \ldots, N-1,\ j = 1, \ldots, M-1, \tag{6.9b}$$

$$v_{i+1,0} = \exp\left(\frac{1}{2}(q-1)a + \frac{1}{4}(q-1)^2 t_{i+1}\right), \quad v_{i+1,M} = 0, \quad i = 0, \ldots, N-1. \tag{6.9c}$$

let $V^{(i)} := (V_{i,1} \cdots V_{i,m})$. linear system:

$$v^{(i+1)} = Av^{(i)}, \quad A := \begin{pmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & 1-2\lambda \end{pmatrix}. \tag{6.10}$$

For the system to be stable:

Thm. If $\Delta t \leq \frac{1}{2}(\Delta X)^2$. Then the finite difference method is stable and converge with error $= O(\Delta t) + O(\Delta X^2)$ if boundary cond. are exact.

Rmk: Let $N \sim M^2$. Then error $\sim M^{-2}$ and comp. cost $\sim M^3$.

If in n-dim, then the cost $\sim M^{2+n}$. (generally let $n \leq 4$).

② **Implicit case**:

Use D_t^-, D_x^2:

$$V_{i-1,j} = V_{i,j} + \lambda(-V_{i,j+1} + 2V_{i,j} - V_{i,j-1})$$

$$Av^{(i)} = v^{(i-1)}, \quad A := \begin{pmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & 1+2\lambda \end{pmatrix}.$$

Thm. It's uncond. stable for $\Delta t > 0$ with error $= O(\Delta t^2) + O(\Delta X^2)$. if BC is exact.

② Crank - Nicolson:

use $D_t^- |_{(t_{i+1}, x_j)}$ and $\frac{1}{2}(D_x^2 |_{(t_i, x_j)} + D_x^2 |_{(t_{i+1}, x_j)})$:

$$\frac{v_{i+1,j} - v_{i,j}}{\Delta t} = \frac{1}{2\Delta x^2}\left(v_{i,j+1} - 2v_{i,j} + v_{i,j-1} + v_{i+1,j+1} - 2v_{i+1,j} + v_{i+1,j-1}\right).$$

And the linear system is:

$$Av^{(i+1)} = Bv^{(i)}, \tag{6.16}$$

where

$$A := \begin{pmatrix} 1+\lambda & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & 1+\lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \cdots & 0 & -\frac{\lambda}{2} & 1+\lambda \end{pmatrix}, B := \begin{pmatrix} 1-\lambda & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & 1-\lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \cdots & 0 & \frac{\lambda}{2} & 1-\lambda \end{pmatrix} \tag{6.17}$$

Thm. If the sol. u given BC $\in C^4$. Then the method is stable for $\forall \Delta x, \Delta t$ with error $O(\Delta t^2) + O(\Delta x^2)$.

(3) Finite element method:

It's more appliable on time-indept PDEs.

Consider variation problem: $A(u,v) = L(v)$ on V where A is sym. positive semidefinite bi-linear form. i.e. $A(v,v) \geq \alpha \|v\|_V^2$, $|A(u,v)| \leq C\|u\|_V\|v\|_V$

Lem. (Minimization problem)

u solves the variation problem $\Leftrightarrow$ u is minimizer of $F(v) := \frac{1}{2} A(v,v) - L(v)$

Pf: ($\Leftarrow$) $\forall v \in W$. assume $v = u + \varepsilon w$. $\varepsilon \in \mathbb{R}^1$.

Then: $F(v) = F(u+\varepsilon w) =: g(\varepsilon)$

$\partial/\partial\varepsilon\, g(\varepsilon) = 0 \Leftrightarrow A(u,w) = L(w)$. $\forall w \in V$.

$\Leftrightarrow$ u solves the problem.

($\Rightarrow$) $F(v) = F(u+\varepsilon w)$

$$= (\tfrac{1}{2} A(u,u) - L(u)) + \varepsilon (A(u,w) - L(w)) + \tfrac{1}{2}\varepsilon^2 A(w,w) \geq F(u).$$

Rmk: It offers a diff. method to prove Lax-Milgram Thm: let $(v_n) \subset V$. $F(v_n) \to \inf_v F(v)$ prove it's Cauchy

To apply FEM:

1) we derive its variation formulation by partial integrate on V ($:= H_0^1$)

2) proj. on finite-dim V_h ($:=$ space of piece wise linear conti. func's).

3) Derive basis $(\phi_i)_1^n$ ($:=$ hat func's) of V_h

Find $u_h \in V_h$. $u_h = \sum_1^N \xi_i \varphi_i$ solve $A(u_h, \varphi_i) = L(\varphi_i)$. $\forall i \leq N$. which's a linear system.

Rmk: To prove the exist & unique of the formulation. We can apply Lax-Milgram's

Thm. $\pi : V \to V_h$ is some proj. Then: we have

$$\|u - u_h\|_V \leq \sqrt{\frac{c}{\alpha}} \|u - \pi u\|_V$$

Pf: By def: $A(u - u_h, v) = 0$. $\forall v \in V_h$.

Set $\|v\| := (A(v, v))^{\frac{1}{2}}$. $e = u - u_h \perp V_h$

$$\Rightarrow \|e\|^2 = A(e, u - \pi u) + A(e, \pi u - u_h)$$

$$= A(e, u - \pi u) \leq \|e\| \|u - \pi u\|.$$

So: $\|e\|_V^2 \leq \frac{1}{\alpha} \|e\|^2 \leq \frac{1}{\alpha} \|u - \pi u\|^2$

$$\leq \frac{c}{\alpha} \|u - \pi u\|_V^2$$

Lem. $V := H_0^1(0,1)$. $V_h := \{ v \in C[0,1] \mid v|_{[x_i, x_{i+1}]}$ is linear. $\forall i$. $v(0) = v(1) = 0$. $x_i = ih$. $h = 1/N \}$.

$\pi : V \to V_h$. $\pi v := \sum_1^N v(x_i) \varphi_i(x)$. where (φ_i) is basis of V_h. Then: $\forall v \in H_0^2(0,1)$

$\Rightarrow \|v - \pi v\|_V \leq ch \|u''\|_{L^2}$.

Pf: It's proved in PDE I.

Rmk: Consider $(-a(x)u'(x))' + r(x)u(x) = f(x)$

$$u(0) = u(1) = 0.$$

If $a \in C^1$, $f \in L^2$. Then we in fact can prove: $\|u\|_{H^2} \le \|f\|_{L^2}$.

Cor. (Aubin-Nitsche duality)

Under cond. of Lem. above

$\Rightarrow \|u - u_h\|_{L^2} \le C h^2 \|u\|_{H^2}$.

Pf: Note $\exists \varphi \in V$. s.t. $\forall v \in V$.

$A(\varphi, v) = \langle e, v \rangle$. Since $e \in H_0^1$

So $e \in L^2 \Rightarrow \varphi$ is well-def.

$$\|e\|_{L^2}^2 = \langle e, e \rangle_{L^2} = A(\varphi, e)$$

$$= A(\varphi - \mathcal{I}\varphi, e)$$

$$\le \|e\| \, \|\varphi - \mathcal{I}\varphi\|$$

By Lem. and Rmk above:

$$\|\varphi - \mathcal{I}\varphi\| \le Ch \|\varphi\|_{H^2} \le Ch \|e\|_{L^2}$$

So: $\|e\|_{L^2}^2 \le Ch\|e\| \le Ch\|e\|_{H^1}$

$$\le C h^2 \|u\|_{H^2}.$$

Next, we consider FEM for parabolic PDE with Dirichlet boundary cond.:

(6.29) $\forall 0 < t \leqslant T, \forall v \in H_0^1(G): \langle \partial_t u(t,\cdot), v\rangle_{H^{-1}(G);H_0^1(G)} + A(u,v;t) = L(v;t), \quad u(0,\cdot) = u_0.$

Assume sol. $u \in L^2([0,T], H_0^1(G))$ exists in the sense of $\partial_t u \in L^2([0,T], H^{-1}(G))$. $G = (0,1)$.

<u>Rmk:</u> i) let $A(u,v) = \langle u', v'\rangle_{L^2}$. $L = 0$. $\Rightarrow$ It becomes heat equation.

ii) It's natural to assume $\partial_t u(t,\cdot) \in H^{-1}(G)$. Since $\partial_t u = \Delta u$ in i). $\Rightarrow$ $1-2=-1$ is order of $\partial_t u$.

For simplicity:

(6.28a) $\quad \partial_t u(t,x) = \Delta u(t,x) + f(t,x), \quad 0 < x < 1, \quad 0 < t \leqslant T,$

(6.28b) $\quad u(0,x) = u_0(x), \quad 0 \leqslant x \leqslant 1, \quad u(t,0) = u(t,1) = 0, \quad 0 \leqslant t \leqslant T,$ (6.28)

where $\Delta = \partial_x^2$ only acts on the space variable x.

<u>Thm.</u> (Energy Dissipation)

$\exists$ const. k. s.t. sol. u of (6.28) satisfies: $\|u(t,\cdot)\|_{L^2}^2 \le e^{-kt}\|u_0\|_{L^2}^2 + k^{-1}\int_0^t e^{-k(t-s)} \|f(s,\cdot)\|_{L^2}^2 ds$

<u>Pf:</u> Note $A(u(t,\cdot), u(t,\cdot))$

$= \|\partial_x u(t,\cdot)\|_{L^2}^2 \overset{\text{Poincaré}}{\geq} 2\|u(t,\cdot)\|_{L^2}^2$

So: $\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 + 2\|u(t,\cdot)\|_{L^2}^2$

$= -A(u(t,\cdot), u(t,\cdot)) + 2\|u(t,\cdot)\|_{L^2}^2 + \int fu$

$\le \int fu \le \|f(t,\cdot)\|_{L^2}\|u(t,\cdot)\|_{L^2}$

$\leq \frac{1}{2}(\|f(t,\cdot)\|_{L^2}^2 + \|u(t,\cdot)\|_{L^2}^2)$

So: $\frac{d}{dt}(e^{kt}\|u(t,\cdot)\|_{L^2}^2) \leq \|f(t,\cdot)\|_{L^2}^2$.

Next, we still consider $h = 1/N+1$. $x_j = jh$.

$\Delta t = T/M$. $t^m = m\Delta t$. Denote: $W^m := W(t^m,\cdot)$.

<u>Def</u>: i) $W^{m+\theta} := \theta W^{m+1} + (1-\theta)W^m$. $0 \leq m \leq M-1$.

ii) Consider $u_h^m \in V_h$. $u_h^0 = \mathrm{Proj}_{V_h}^{L^2} u_0$. We say $(u_h^m)_{m=0}^M$ is sol. of θ-scheme if: $\forall 0 \leq m \leq M-1$.

$$\left\langle \frac{u_h^{m+1} - u_h^m}{\Delta t}, v \right\rangle_{L^2} + A(u_h^{m+\theta}, v) = \langle f^{m+\theta}, v\rangle_{L^2}$$

For $\theta = 0$. it's forward Euler approxi.

For $\theta = 1$. it's backward Euler approxi.

<u>Thm</u>. For $1/2 \leq \theta \leq 1$. θ-scheme is uncond. stable. i.e. $\max_m \|u_h^m\|_{L^2}^2 \leq \|u_h^0\|_{L^2}^2 + \Delta t \sum_0^{M-1} \|f^{m+\theta}\|_{L^2}^2$

For $0 \leq \theta < 1/2$. θ-scheme is stable if $\exists 0 < \varepsilon < 1$. $\Delta t \leq h^2(1-\varepsilon)/(6(1-2\theta))$. Besides, $\max_m \|u_h^m\|_{L^2}^2 \leq \|u_h^0\|_{L^2}^2 + C_\varepsilon \Delta t \sum_0^{M-1} \|f^{m+\theta}\|_{L^2}^2$. where $C_\varepsilon = (4\varepsilon^2)^{-1} + \Delta t(1-2\theta)(1+\varepsilon^{-1})$.

Pf: Only for $1/2 \le \theta \le 1$.

Note $u_h^{m+\theta} = \Delta t(\theta - \frac{1}{2}) \frac{u_h^{m+1} - u_h^m}{\Delta t} + \frac{u_h^m + u_h^{m+1}}{2}$

Plug it into the scheme:

$$0 \le \Delta t(\theta - 1/2)\left\|\frac{u_h^{m+1} - u_h^m}{\Delta t}\right\|_{L^2}^2 + \frac{1}{2\Delta t}\left(\|u_h^{m+1}\|_{L^2}^2 - \|u_h^m\|_{L^2}^2\right) + \|\nabla u_h^{m+\theta}\|_{L^2}^2 = \langle f^{m+\theta}, u_h^{m+\theta}\rangle_{L^2} \le \square + \square$$

Thm. For u is sol. of (6.28) and $(u_h^m)_0^M$ is its backward Euler approx.: If $\sup_{[0,T]} \|\partial_t^2 u(t,\cdot)\|_{L^2} \vee \sup_{[0,T]} \|\partial_t u(t,\cdot)\|_{H^2} < \infty$.

Then: $\exists C > 0$. st. $\max_{m=1,\dots M} \|u^m - u_h^m\|_{L^2} \le C(\Delta t + h^2)$

Pf: Denote $\mathrm{Proj}_h : V \to V_h$. $\|v\| = (A(v,v))^{1/2}$

Error decomp. $e_h^m := u^m - u_h^m := \eta^m + \xi^m$

$\eta^m := u^m - P_h u^m \in V$. $\xi^m = P_h u^m - u_h^m \in V_h$.

1) By Lem. before: $\|\eta^m\|_{L^2} \le Ch^2 \|u^m\|_{H^2}$

Similarly. $\left\|\frac{\eta^{m+1} - \eta^m}{\Delta t}\right\|_{L^2} \le Ch^2 \left\|\frac{u^{m+1} - u^m}{\Delta t}\right\|_{H^2}$

2) Next we estimate ξ^m:

$m = 0$. Note $u_h^0 = \mathrm{Proj}^{L^2}_{V_h} u^0$. ξ_0:

$e_h^0 \perp^{L^2} V_h \Rightarrow \langle \xi^0, v\rangle_{L^2} = -\langle \eta^0, v\rangle_{L^2}$

let $v = \xi^0 \in V_h \Rightarrow \|\xi^0\|_{L^2} \le \|\eta^0\|_{L^2} \le Ch^2 \|u_0\|_{H^2}$

For $1 \le m \le M$. we have for $\forall v \in V_h$.

$$\left\langle \frac{\xi^{m+1}-\xi^m}{\Delta t}, v \right\rangle_{L^2} + A(\xi^{m+1}, v) = \left\langle \frac{u^{m+1}-u^m}{\Delta t} - \partial_t u^{m+1} - \frac{\eta^{m+1}-\eta^m}{\Delta t}, v \right\rangle_{L^2}$$

$$\stackrel{\Delta}{=} \langle \tilde{f}^{m+1}, v \rangle_{L^2}$$

from backward Euler scheme.

By Thm above:

$$\max \|\xi^m\|_{L^2}^2 \le \|\xi^0\|_{L^2}^2 + \Delta t \sum_0^{m-1} \|\tilde{f}^{m+1}\|_{L^2}^2.$$

i) Estimate $\|\tilde{f}^{m+1}\|_{L^2}^2$:

$$\|\tilde{f}^{m+1}\|_{L^2} \le \left\| \frac{u^{m+1}-u^m}{\Delta t} - \partial_t u^{m+1} \right\|_{L^2} + \left\| \frac{\eta^{m+1}-\eta^m}{\Delta t} \right\|_{L^2}$$

$$:= A + B.$$

$$B \stackrel{1)}{\le} Ch^2 \|\Delta t^{-1} \int_{t^m}^{t^{m+1}} \partial_t u(t,\cdot)\,dt\|_{H^2}$$

$$\stackrel{\text{Hölder}}{\le} Ch^2 \left(\int_{t_m}^{t_{m+1}} \|\partial_t u(t,\cdot)\|_{H^2}^2 dt\right)^{\frac{1}{2}} / \Delta t^{\frac{1}{2}}.$$

$$A \stackrel{\text{Taylor}}{=} \|-\Delta t^{-1} \int_{t_m}^{t_{m+1}} (t-t^m)\,\partial_t^2 u(t,\cdot)\,dt\|_{L^2}$$

$$\left|\int_{t_m}^{t_{m+1}} (t-t^m)\,\partial_t^2 u(t,\cdot)\,dt\right|^2 \le$$

$$\int_0^1 dt \int_{t_m}^{t_{m+1}} (t-t^m)^2 (\partial_t^2 u(t,\cdot))^2 dt$$

$$\le \Delta t^2 \int_{t_m}^{t_{m+1}} (\partial_t^2 u(t,\cdot))^2 dt.$$

$$\Rightarrow A^2 \le C\Delta t \int_{t_m}^{t_{m+1}} \|\partial_t^2 u(t,\cdot)\|_{L^2}^2.$$

<u>Rmk</u>: For $\theta = \frac{1}{2}$ - scheme. L^2-error $= \Delta t^2 + h^2$