Preliminary

① <u>Remark</u>: i) $\bar{S} = (S_t^0, S_t)_{t\in[0,T]}$. càdlàg. SP describes price evolution of the numéraire $S^0 > 0$ and risky asset S.

ii) $\bar{X} = (X^0, X) := S/S^0 = (1, S_t/S_t^0)$ the discounted price.

iii) $\bar{\xi} = (\xi_t^0, \xi_t)_{[0,T]}$. predictable SP. describes investment strategy.

And $V^{\bar{\xi}} := \bar{\xi}\bar{X}$. discounted wealth

<u>Rmk</u>: $\bar{\xi}$ is predictable means it can be approxi. by simple strategies: $\hat{\sum_i} \eta_i I_{(T_{i-1}, T_i]}(t)$ where $\eta_i \in \mathcal{F}_{T_{i-1}}$

iv) $V_t^{v,\xi} = v + \int_0^t \xi_s \, dX_s \triangleq v + G_t(\xi)$.

$G_t(\xi)$ is gains by time t with strategy ξ.

Rmk: $G_t(\xi)$ is extended from $\xi = \sum \eta_i I_{(t_{i-1}, t_i]}$. $\eta_i \in \mathcal{F}_{t_{i-1}}$. With the Lemma below for pointwise t.

(Itô integral def in L^2-limit is not pointwise. will depend on $\mathbb{P}$)

Lemma. (ξ_t^n) simple $\to \xi$. in $L^\infty(\mathbb{P}) \Rightarrow G_t(\xi^n) \xrightarrow{pr} G_t(\xi)$. $\forall t \in [0, T]$ iff the adapted càdlàg process X is Semimart.

Rmk: If we require (ξ_t^n) be arbitrary bdd seq. i.e. $|\xi_t^n| \leq 1$. $\xi_t^n \xrightarrow{L^\infty} \xi_t$. Thm. X must be FV.

So we def $\int \xi dX$ from simple func.

Rmk: No arbitrage (NFLVR). i.e.

$\overline{L^\infty \{ g \in L^\infty \mid g \geq 0.\ \exists \xi \text{ simple. } G_T(\xi) \geq g \}} = \{0\}$.

Can also imply that X is a Semimart. without addressing the stability above. from Girsanov Thm.

Next, we assume $(\mathcal{F}_t)$ is complete.

<u>Lemma.</u> $\forall$ local mart. M_t can have a unique modification $\tilde{M}_t$. with càdlàg path. i.e. $\mathbb{P}(M_t = \tilde{M}_t) = 1$. $\forall t$.

<u>Rmk</u>: $\{M_t = \tilde{M}_t . \forall t\}$ may not be measurable! (uncountable state).

<u>Pf</u>: Actually any mart. has a càdlàg modification if $(\mathcal{F}_t)$ is complete.

Find $(T_n) \nearrow \infty$. Note $\cup \{T_n > t\} = 1$.

$\tilde{M}_t := \tilde{M}_{t \wedge T_n}$. if n satisfies $T_n > t$.

Uniqueness if from consider $\mathbb{Q} \cap [0, T]$.

So next, we want to model the asset price as semimart : $X = M + A$.

i) A is FV. process. i.e. average price changed over next dt - period.

ii) M is local mart. specify price fluctuations.

<u>Denote</u>: $L(X) \triangleq \{ \int_0^\cdot \xi_s^2 \, d[M]_s + \int_0^\cdot |\xi_s| \, |dA_s| \overset{a.s.}{<} \infty \}$.

① Quadratic Variation:

Recall in construction of QV of c.l.m.

We hand on a "iterated intergral": $\int_0^t M_s\,dM_s$

$= \frac{1}{2}M_t^2 - \frac{1}{2}[M]_t$. i.e. We consider:

$\lim_{n\to\infty} \frac{1}{2}(M_t^2 - \sum_i (M_{t_i^n} - M_{t_{i-1}^n})^2)$. Note that

$M_t^2 = (\sum_i (M_{t_i^n} - M_{t_{i-1}^n}))^2$. We have:

I_t equals: $\sum_i M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n}) \triangleq X_t^n$.

Then: We check X_t^n is L^2-Cauchy. $\Rightarrow$

I_t has L^2-limit. Since $X_t^n = I_t^M(g^n)$. s.t.

$g_s^n = \sum M_{t_{j-1}^n} I_{(t_{j-1}^n, t_j^n]}(s)$. $\Rightarrow$ X_t^n is mart.

So: $X_t^n \xrightarrow{L^2} X_t$ is mart. And limit above exist

For c.l.m. case: Set $[M]_t := [M]_{t\wedge T_n}$ on $[t \le T_n)$

<u>Rmk</u>: 1) $\langle M\rangle$. $[M]$ coincides if M is conti.

But they have some difference:

$\langle M\rangle_t$ is from Doob-Meyer's decompose

i.e. $M_t^2 - \langle M\rangle_t \in \mathcal{M}^{loc}$ for $M \in \mathcal{M}^{loc}$.

It can also def on some merely càdlàg local mart, which's called predictable QV.

While $[X]_t$ is characterized by:

$X_t^2 - [X]_t = 2\int_0^t X_{s-}\, dX_s$. And it's not predictable generally. (But both càdlàg)

And for (càdlàg) mart $X_t = X_t^{\text{conti}} + X_t^{\text{discrete}}$.

$[X]_t = [X^c]_t + [X^d]_t$

$= \langle X^c \rangle_t + \sum_{s<t} |\Delta X_s|^2$.

iii) The convergence of $\sum (M_{t_i^n} - M_{t_{i-1}^n})^2 - [M]_t$ can also hold in ucp. sense.

⑧ <u>Stochastic exponential:</u>

Recall for SDE: $dX_t = X_t\, dL_t$. $X_0 = x_0$.

if we consider $Y_t = x_0 e^{L_t}$. $\Rightarrow$ Apply Itô's

$dY_t = Y_t (dL_t + \frac{1}{2} d[L]_t)$.

So we need a compensator on exp.:

$\mathcal{E}(L)_t = \exp(L_{0,t} - \frac{1}{2}[L]_t)$. $\Rightarrow X = x_0 \mathcal{E}(L)$

<u>Rmk</u>: L_t is called stochastic logrithm. which

can be uniquely chosen by $L_t = \int_0^t \frac{dX_s}{X_s}$.

$\Rightarrow$ Set $L_t = \sigma W_t + (\mu - r)t$. We have the discounted Black-Scholes model:

$dX_t = X_t((\mu - r)dt + \sigma dW_t)$

④ Mart. measure:

i) Jomar in la mart.:

Consider the strategy performed inductively on $[0, T]$:

$n=1$: Set $f_1 = 1$. (hold one stock). until gain 1 \$. but at most wait until $t = \frac{T}{2}$.

From n to $n+1$: Once already gain 1 \$. then leave the game. Otherwise if it doesn't happen by the time $= T_n \overset{\Delta}{=} (1-2^{-n})T$

Choose $(n+1)^{th}$ strategy $f_{n+1} \in \mathcal{F}_{T_n}$. st.

$$\mathbb{P}\Big(\sup_{[T_n, T_{n+1}]} (V_{T_n} + f_{n+1}(X_t - X_{T_n})) \geq 1 \,\Big|\, \mathcal{F}_{T_n}\Big) \geq 1/2.$$

Rmk: It's possible since if we set $\xi_{n+1} = X \cdot \in \mathbb{R}'$. Note that

$$\mathbb{P}\left(\frac{1-V_{T_n}}{\sup_{[T_n,T_{n+1}]} X_t - X_{T_n}} \le X \mid \mathcal{F}_{T_n}\right) \uparrow 1. \quad (X \to \infty)$$

(It's not reasonable if no fluctuation on X_t. i.e. $\forall X_t = X_{T_n}$)

Lemma. The SF strategy ξ above yields $V_T = 1$. a.s. And it can be completed by finite many operations a.s.

Pf: $\mathbb{P}(V_T < 1) = \mathbb{P}(\bigcap_n \{\text{unlucky at } t = T_n\})$

$$= \lim_N \mathbb{E}\Big(\mathbb{P}(\text{unlucky at } t = T_N \mid \mathcal{F}_{T_{N-1}}) \cdot \prod_1^{N-1} I(\text{unlucky at } t = T_k)\Big)$$

$$\le \lim_N \frac{1}{2} \mathbb{E}\Big(\prod_1^{N-1} \cdots\Big) \cdots \le \lim_N 2^{-N} = 0.$$

And $\sum_n \mathbb{P}(\text{unlucky at } t = T_n) \le$

$$\le \sum_n 2^{-n} < \infty. \Rightarrow \mathbb{P}(\cdots, i.o) = 0.$$

Rmk: i) It's called doubling strategy which can produce riskless profit.

But it's not admissible strategy

(i.e. $\exists c > -\infty$. s.t. $\int_0^t \xi_s \, dX_s \overset{a.s.}{\geq} c$)

ii) Admissible isn't needed in discrete time because it only performs finite operations in finite time

iii) It's origin of martingale.

Thm. Local mart. p.m. $\mathbb{P}^* \sim \mathbb{P}$. exists. $\iff$

(i.e. X_t is local mart. under $\mathbb{P}^*$).

$\overline{L^-} \{ g \in L^{\infty}_+ \mid g \leq \int_0^T \xi_s dX_s . \exists \xi \in L(X) \text{ adm} \}$

$= \{0\}$. i.e. NFLVR.

To find the risk-neutral p.m. $\mathbb{P}^*$:

Set $Z_t := d\mathbb{P}^*/d\mathbb{P} |_{\mathcal{F}_t}$ if it exists (then

Z_t will be u.i. mart.)

S. : X is $\mathbb{P}^*$-local mart $\iff$ XZ is $\mathbb{P}$-□.

By Itô's : $d(XZ) = X dZ + Z dX + d\langle X, Z \rangle$

$= X dZ + Z dM + (Z dA + d\langle X, Z \rangle)$

$\Rightarrow$ We need : $Z dA = -d\langle X, Z \rangle$.

Note $Z_t \geq 0$. Assume $Z_t = \mathcal{E}(L)$.

So: $dA = -dX \cdot dZ / Z = -dX \cdot dL = -d\langle X, L\rangle$

which is necessary for $XZ \in \mathcal{M}^{l.c}_{c}$.

<u>Rmk</u>: i) (Sufficiency)

We see if such Z_t exists. then:

$X = M - \langle X, L\rangle = M - \langle M, L\rangle \in \mathbb{P}^* - \mathcal{M}^{l.c}_{c}$.

where we can set $\mathbb{P}^*(A) = E_{\mathbb{P}}(I_A Z_\infty)$

ii) Note $\{\mathbb{P}-\text{Semimart}\} = \{\mathbb{P}^* - \text{Semimart}\}$.

Since $M + A = M - \langle M, L\rangle + (A + \langle M, L\rangle)$

$\Rightarrow L(X)$ under $\mathbb{P}$ $= L(X)$ under $\mathbb{P}^*$

iii) From i): After constructing Z, we need to check it's true mart. to get $\mathbb{P}^* \sim \mathbb{P}$.

Apply on Black-Scholes model:

$dX_t / X_t = \sigma_t dW_t + (\mu_t - r_t) dt$.

Set $Z_t = \mathcal{E}(L)_t$. Then:

$d[X, L]_t + X_t(\mu_t - r_t) dt = 0$.

$\Rightarrow d[X, L]_t = \sigma_t d[W, L]_t = -X_t(\mu_t - r_t)(dW_t)^2$.

So: Set $L_t = -\int_0^t \frac{\mu_s - r_s}{\sigma_s} dW_s$

$$\stackrel{\Delta}{=} -\int_0^t \theta_s \, dW_s$$

We require Novikov condition:

$\mathbb{E}\left(\exp\left(\frac{1}{2} \int_0^T |\theta_s|^2 ds \right)\right) < \infty$ holds.

$\Rightarrow Z_t = \mathcal{E}\left(-\int_0^t \theta_s \, dW_s \right)$ is the density which yields EMM

<u>Rmk</u>: generally, for $L \in \mathcal{M}_{loc}^c$, $L_0 = 0$. Then:

$\mathcal{E}(L) \in \mathcal{M}_c^{loc}$ with $\mathcal{E}(L) \geq 0$. So:

$\mathcal{E}(L)$ is supermart. And $\mathcal{E}(L)$ is mart. on $[0,T] \Leftrightarrow \mathcal{E}(L)_T = 1$.

<u>Pf</u>: $\mathbb{E}(\mathcal{E}(L)_0) = 1 \geq \mathbb{E}(\mathcal{E}(L)_t) \geq \underbrace{\mathbb{E}(\mathcal{E}(L)_T)}_{=1}$

So: $\mathbb{E}(\mathbb{E}(\mathcal{E}(L)_t \mid \mathcal{F}_s)) = \mathbb{E}(\mathcal{E}(L)_s)$.

<u>Cor</u>. (Exponential inequi. for $\mathcal{M}_c^{loc}$)

$$\mathbb{P}\left(\sup_{[0,t]} (M_s - q\langle M\rangle_s) \geq k \right) \leq e^{-2qk}$$

<u>Pf</u>: $LHS = \mathbb{P}\left(\max_{[0,t]} \mathcal{E}(2qM.)_s \geq e^{2qk} \right)$

$$\underset{\substack{\text{supermart.} \\ \geq 0}}{\overset{\text{Doob's}}{\leq}} e^{-2qk} \, \mathbb{E}\left(\mathcal{E}(2qM)_0 \right) = e^{-2qk}$$

Rmk: Although for supermart/submart. We always think on M_t^- / M_t^+. It just for retaining its mart prop. by X^- / X^+. The Doob's inequs. also work for supermart. $M \geq 0$:

$$\lambda \mathbb{P}(\max_{[0,t]} M_s \geq \lambda) \leq \mathbb{E}(M_0).$$

Also for submart. M:

$$\lambda \mathbb{P}(\max_{[0,t]} M_s \geq \lambda) \leq \mathbb{E}(M_t^+)$$

Cor. (Bernstein's inequi.)

$M \in \mathcal{M}_c^{loc}$. $M_0 = 0$. Then we have:

$$\mathbb{P}(M_\infty^* \geq x, \langle M \rangle_\infty \leq y) \leq e^{-x^2/2y}.$$

Pf: $LHS \leq \mathbb{P}(\sup(M_t - \lambda \langle M \rangle_t) + \lambda \langle M \rangle_\infty \geq x, \square)$

$$\leq \mathbb{P}(\sup(M_t - \tfrac{\lambda}{2} \langle M \rangle_t) \geq x - \tfrac{\lambda y}{2})$$

$$= \mathbb{P}(\sup \mathcal{E}(\lambda M) \geq e^{\lambda(x - \frac{\lambda y}{2})})$$

$$\overset{Doob}{\leq} e^{-\lambda(x - \lambda y/2)} \mathbb{E}(\mathcal{E}(\lambda M)_0)$$

choose $\lambda = \dfrac{x}{y}$ to optimize