Hedging and Pricing

· **Hedging**: It's to apply some strategy to offset loss or gain. Replicate is one of hedging method and it's the best. Since it's totally risk-free.

e.g. Holding some derivative. Replicating is to find a portfolio of other assets, which totally imitates the behavior of derivative.

(1) **Arbitrage-free valuation**:

X is a discounted price for some risky asset and $\mathbb{P}^*$ is its equi. local mart. p.m.

Consider contingent claim with discounted payoff $H \geq 0$. at $t = T$.

Def: Arbitrage-free valuation X^H for the claim with initial price x is c.l.m. under p.m. $\mathbb{P}^*$. And $X_0^H = x$. $X_T^H = H$.

If $H \in L(\mathbb{P}^*)$. We can set $X_t^H := E^*(H|\mathcal{F}_t)$.
as its arbitrage-free valuation.

<u>Lemma.</u> (cheapest price won't change if hedgable)

If $H \geq 0$. $H \in \mathcal{F}_T$. $H \in L(\mathbb{P}^*)$. s.t. $\forall t$

$X_t^H \triangleq \mathbb{E}^*(H|\mathcal{F}_t)$ is replicable by X_t:

$X_t^H = z^H + \int_0^t \xi_s^H dX_s$. for some $z_H \in \mathbb{R}^1$.

and $\xi^H \in L(X)$. Then: $z^H = \inf\{z \in \mathbb{R}^1 |$

$\exists$ adm. $\xi \in L(X)$. s.t. $z + \int_0^T \xi_s dX_s \geq H$. a.s$\}$

Besides. X^H doesn't depend on choice

of equi. measure $\mathbb{P}^*$. if ξ^H exists.

<u>Rmk</u>: i) If the NA price $\mathbb{E}^*(H|\mathcal{F}_t)$ can

be replicated by other asset

X. Then different EMM $\widetilde{\mathbb{P}}^*$

won't lead to different price.

ii) Consider "suicide strategy" $\tilde{\xi}$. i.e.

"doubling strategy" ends in -1. a.s.

rather. It's a admissible strategy

But set $\tilde{z} = E^*(H) + 1$. And

Note that $\xi^H + \tilde{\xi}$ also replicates

claim $H = \tilde{z} + \int_0^T (\xi^H + \tilde{\xi})\, dX$. $\Rightarrow$

it produces a higher price $\tilde{z}$ at

$t = 0$. (Repli. H alone $\not\Rightarrow$ pin down AFP)

Pf: Note for $\forall z \in [\cdot\cdot]$. ξ. ad. s.t. $z + \int_0^T \cdots$

$\Rightarrow z + \int_0^t \xi_s\, dX_s$ is P^*-supermart. $\geq H$

(consider $z + \int_0^t \xi_s dX_s - E^*(H \mid \mathcal{G}_t) \geq 0$)

So: $z \geq E^*(z + \int_0^T \xi\, dX)$

$\geq E^*(H) = z^H$

Since z_H is characterized by $\inf[\cdots]$.

a.s. $\Rightarrow$ it's indept of choice of Emm.

X_t^H can also be characterized by that

$X_t^H = \inf[X_t \in \mathcal{G}_t \mid \exists$ adm $\xi \in L(X)$. s.t.

$X_t + \int_t^T \xi\, dX \geq H$. a.s.$]$. follows from

$X_t + \int_t^s \xi\, dX \overset{\Delta}{=} M_s$, $s \geq t$ is a P^*-supermart.

$X_t \geq E^*(M_T \mid \mathcal{G}_t) \geq E^*(H \mid \mathcal{G}_t)$

(2) <u>Kunita-Watanabe Decomp.</u>:

Note that we can't hedge by replicating

all the time. Sometimes we will super-replicate it by trading X. (i.e. find strategy ξ s.t. $z + \int \xi \, dX \geq H$). But it's too costly and not fair to charge super-replicate price!

Next, set $X \in \mathcal{M}^2(\mathbb{P}^*) := \{$càdlàg L^2-bdd mart on $[0,T]\}$ under $\mathbb{P}^*$.

<u>Thm.</u> (Kunita-Watanabe Decomp.)

$\forall (H_t) \subset \mathcal{M}^2(\mathbb{P}^*)$. We have a unique decomp

$$H_t = H_0 + \int_0^t \xi_s \, dX_s + N_t,\quad t \in [0,T]. \text{ s.t.}$$

i) ξ^H is predictable. $\mathbb{E}^*\left(\int_0^T (\xi_s^H)^2 \, d[X]_s\right) < \infty$

ii) $N \in \mathcal{M}^2(\mathbb{P}^*)$. $N_S \perp X_S$. for $\forall$ stopping time $S \leq T$. ($\mathbb{E}^*(N_S X_S) = 0$))

iii) $H_0 = \mathbb{E}^*(H_T)$.

<u>Rmk</u>: At least, we can find a "optimal" hedging strategy.

<u>Pf</u>: WLOG. Set $H_0 = 0$. $X_0 = 0$.

$$\Gamma := \left\{ \int \xi \, dX \;\middle|\; \xi \in L^2_{pred}(\mathbb{P}^* \otimes d[X].) \right\}$$

For $\int \xi^n dX \subset \Gamma \xrightarrow{L^2} \int \xi \, dX$. Apply Itô's:

$\mathbb{E}^*(\int|\xi^n-\xi|^2 d[X]_s) \to 0$. which's equi: $\xi^n \xrightarrow{L^2} \xi$. $\Rightarrow \xi \in L^2_{pred}(\mathbb{P}^* \otimes d[X]_s)$.

So: $\Gamma \subseteq_{close} \mathcal{M}^2(\mathbb{P}^*)$. Γ is Hilbert space

$\forall M_t \in \mathcal{M}^2(\mathbb{P}^*)$. We have $Proj_\Gamma M_t \triangleq M_t^\Gamma$

Set $N_t \triangleq M_t - M_t^\Gamma \perp M_t^\Gamma$.

$\Rightarrow \mathbb{E}^*(N_T \int_0^T \xi_t \, dX_t) = 0$. Replace ξ_t by $\xi_t I_{[0,s]}(t)$. So: $\mathbb{E}^*(N_T \int_0^s \xi \, dX) = 0$.

With mart. prop. of N. $\Rightarrow \mathbb{E}^*(N_s \int_0^s \square) = 0$.

For uniqueness: $(\tilde{N}, \tilde{\xi})$ satisfies $\square$

$\Rightarrow \int \xi - \tilde{\xi} \, dX = \tilde{N} - N \perp \Gamma$. So: $\tilde{N} = N$.

With $\mathbb{E}^*(\int |\xi-\tilde{\xi}|^2 I_A \, d[X]) = 0 \Rightarrow \xi = \tilde{\xi}$.

<u>Rmk</u>: It also holds for $M, X \in \mathcal{M}_{loc}(\mathbb{P}^*)$ as well. Then: $\exists! \xi \in L^2_{loc,pred}(X)$ and $N \in \mathcal{M}_{loc}(\mathbb{P}^*) \perp X$ strongly.

Note in both case, we didn't have any continuity assumption! (But discrete term in N_t in fact.)

Cor. Set $H_t = E^*(H \mid \mathcal{G}_t)$ above. and get $H_t = H_0 + \int_0^t \xi_s^H dX_s + N_t^H$. Then:

$$\xi^H = \underset{\xi \in L^2_{pred}(P^* \otimes \lambda[X])}{\arg\min} E^*((H - (V_0 + \int_0^T \xi dX))^2)$$

Pf: $= (E^*(H_T) - V_0)^2 + E^*((H_T - H_0 - \cdots)^2)$.

Rmk: i) ξ^H, $E^*(H) = H_0$ will both depend on choice of P^*.

ii) It leaves all the risk on the unhedgable ortho. term N_t.

Lemma (Doob-Dynkins)

$T: (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$ measurable. $f: \Omega \to \mathbb{R}'$. is $B_{\mathbb{R}'}$-measurable. Then: $f \in \sigma(T)$

$\Leftrightarrow \exists g: \Omega' \to \mathbb{R}'$. $B_{\mathbb{R}'}$-measurable. s.t. $f = g(T)$.

Pf: If $f = I_A$, $A \in \mathcal{A}$. Set $g = I_{T(A)}$ ✓.

If $f \geq 0$. We can approxi. it by seq of $\uparrow$ simple seq f_n. s.t. $f_n = g_n \circ T$

$\Rightarrow$ Set $g = \sup_n g_n$ still measurable.

Besides, with $g|_{R(T)}(x) = \lim_n g_n|_{R(T)}(x)$.

$\Rightarrow f = g \circ T$. Converse is trivial.

__Lemma:__ $\forall M$ càdlàg adapted. M is mart. $\Leftrightarrow$

$\forall T$ bdd stopping time. $M_T \in L^1$. $E(M_T) = E(M_0)$

__Pf:__ ($\Leftarrow$) Set $T \equiv t$. $\Rightarrow M_t \in L^1$

Set $T = I_A t + I_{A^c} s$. $A \in \mathcal{F}_t$.

($\Rightarrow$) Optional Sampling Thm

__Thm.__ (Itô's representation)

W is $\mathcal{F}_t$-BM on $(\Omega, (\mathcal{F}_t), P)$.

i) $\forall M \in \mathcal{M}_{loc}(P^?)$. $M_t \in \mathcal{F}_t^W$. $\forall t \geq 0$. $\Rightarrow$

$M_t = M_0 + \int_0^t \xi_s dW_s$. for some $\xi \in L(W)$

ii) $\forall T \geq 0$. $H \in \mathcal{F}_T^W$. $H \in L^2(W)$. Then:

$H = E(H) + \int_0^T \xi_s dW_s$. for some $\xi \in L^2(P \otimes \lambda t)$

iii) ξ in above represent i). ii) is

unique in their class ($L(W)$; $L^2(P \otimes \lambda t)$)

__Rmk:__ i) $\mathcal{F} = \mathcal{F}^W$ is necessary. if $B \perp W$

is Bm. and $B_t = \int_0^t H_s\, dW_s \Rightarrow$

$1 = \mathbb{E}(B_t^2) = \mathbb{E}(\langle B, H\cdot W\rangle_t) = \mathbb{E}(H\cdot\langle B, W\rangle)$
$= 0$

iv) If we consider $L^1(x)$-class in ii):

Set ξ^0, ξ^s is doubling and suicide strategy. St. $\int_0^{\frac{T}{2}} \xi^0\, dW = 1$. $\int_{\frac{T}{2}}^T \xi^s\, dW = -1$.

$\Rightarrow \tilde{\xi} = \xi + \xi^0 + \xi^s$ is another repli. $\neq \xi$ in $L(W)$.

Pf: iii) $\mathbb{E}(H \mid \mathcal{F}_t) = \mathbb{E}(H) + \int_0^t \xi\, dW$

$= \mathbb{E}(H) + \int_0^t \tilde{\xi}\, dW.\ \forall t$

$\Rightarrow$ By Itô's isometry. $\xi = \tilde{\xi}$. a.s.

ii) Set $H^n = -n \wedge H \vee n \xrightarrow{L^2} H$. Note:

By KW decomp. $H^n \in L^2$. We have:

$H_t^n \stackrel{\Delta}{=} \mathbb{E}(H_T^n \mid \mathcal{F}_t^W)$

$= \mathbb{E}(H_T^n) + \int_0^t \xi_s^n\, dW_s + N_t^n$.

Next, we prove: $\|N_T^n\|_\infty = 0$

By contradiction: if $\|N_T^n\|_\infty > 0$:

Set $\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^W} = 1 + c N_T^n / \|N_T^n\|_\infty$. $c > 0$.

Since bdd stopping time S. We have:

$$\widetilde{\mathbb{E}}(W_S) = \mathbb{E}(W_S) + \frac{1}{\|N_T\|_\infty}\mathbb{E}(N_T^n W_S)$$

$$= 0. \Rightarrow W_S \text{ is } \widetilde{\mathbb{P}}\text{-mart.}$$

Besides. $\langle W \rangle_t = t$ $\widetilde{\mathbb{P}}$-a.s. So: W is a $\widetilde{\mathbb{P}}$-BM.

By Factorization Lemma:

$N_T = f^n(W_s, s \le T)$ ($H \in \mathcal{F}_T^W \Rightarrow N_T \in \mathcal{F}_T^W$).

So: $\mathbb{E}(N_T^n) = \mathbb{E}(f^n(W_s, s \le T))$

$$= \widetilde{\mathbb{E}}(f^n(W_s, s \le T))$$

$$= \mathbb{E}(N_T^n) + \frac{1}{2\|N_T\|_\infty}\mathbb{E}((N_T^n)^2)$$

$\Rightarrow N_T^n \equiv 0$ a.s. Contradiction!

So: We have: $H_t^n = \mathbb{E}(H_T^n) + \int_0^t \xi_s^n dW_s$

By Itô's isometry $H_t^n \xrightarrow{L^2} H_t$. $\Rightarrow \exists \xi$.

$\xi_t^n \xrightarrow{L^2} \xi_t$. (first check: Cauchy seq.)

So: $H_t = \mathbb{E}(H_T) + \int_0^t \xi_s dW_s$.

i) First, prove: $M_t \in \mathcal{F}_t^W \Rightarrow M$ is conti. a.s.

Set $M^n = -n \wedge M \vee n \in L^2$. $\forall t \ge 0$.

So by ii) M_t^n is conti.

Note $P(\sup_{[0,T]} |M_t - M_t^n| \geq \varepsilon) \overset{\text{Doob}}{\leq} E(|M_T - M_T^n|)/\varepsilon$

$\overset{\exists (n_k)}{\Longrightarrow} \sup_{t \leq T} |M_t - M_t^{n_k}| \to 0$ a.s. by Borel-Cantelli.

Next, set $T_n = \inf\{t \geq 0 \mid |M_t|^2 > n\}$. $\overset{\text{a.s.}}{\nearrow}$ localization since M_t conti. $\Rightarrow M^{T_n} \in L^2$

has local repre. Let $M_t = M_{t \wedge T_n}$ on $\{T_n \geq t\}$

Recall a market is complete if $\forall$ bdd contigent claim $H \in \mathcal{F}_T^X$. can be replicated by asset X. i.e. $\exists V^H \in \mathbb{R}^1$. $\xi^H \in L(X)$. s.t.

$H = V^H + \int_0^T \xi_s^H dX_s$. and $(\int_0^t \xi_s^H dX_s)$ is bdd.

<u>Cor.</u> Black-Scholes (μ, r, σ) model. driven by BM (W_t). with $\mathcal{F}_t = \mathcal{F}_t^W$. $\theta \triangleq \frac{\mu - r}{\sigma}$.

i) It's complete

ii) $Z_t = \frac{dP^*}{dP}\Big|_{\mathcal{F}_t} := \mathcal{E}(-\theta W)_t$ is the only EMM. for discounted price $X_t = e^{-rt} S_t$

<u>Pf</u>: i) $W_s^* = W_s + \int_0^t \frac{\mu - r}{\sigma} ds \Rightarrow \mathcal{F}_t^{W^*} = \mathcal{F}_t^W$

Apply Ito's representation Thm:

$$H = E^*(H) + \int_0^T \xi_s^H \, dW_s^* = E^*(H) + \int_0^T \frac{\xi_s^H}{\sigma X_s} \, dX_s$$

$$\mathbb{E}(H \mid \mathcal{F}_t^W) = E^*(H) + \int_0^t \xi_s^H / \sigma X_s \, dX_s \Rightarrow \text{hold.}$$

ii) If $\tilde{Z}_t = \frac{d\tilde{\mathbb{P}}^*}{d\mathbb{P}} \Big|_{\mathcal{F}_t^W} = \mathcal{E}(\tilde{L}_t)$ is another

By Repre. Thm. $\tilde{L}_t = \int_0^t \tilde{v}_s \, dW_s$

By necessarity of Girsanov. Thm:

We have: $dA_t + d\langle X, \tilde{L} \rangle_t = 0$

$$dA_t = (\mu - r) X_t \, dt$$

$$dX_t \, d\tilde{L}_t = \sigma X_t \cdot \tilde{v}_s \, dt$$

$$\Rightarrow \tilde{v}_s = \frac{(\mu - r)}{\sigma}.$$

<u>Cor.</u> (Complete in general.)

For a conti. discounted price $(X_t)_{[0,T]}$.

on $(\Omega, (\mathcal{F}_t), \mathbb{P})$. We have:

i) $\exists$ unique equi. local mart. measure $\mathbb{P}^*$ for X on $\mathcal{F}_T$.

ii) X has no arbitrarge and the market is complete.

Then: We have i) $\Leftrightarrow$ ii).

<u>Pf:</u> By FTAP: NA $\Leftrightarrow$ EMM $\mathbb{P}^*$ exists

i) $\Rightarrow$ ii) WLOG. Assume $X \in L^2$.

(Otherwise, we can consider $\widehat{X_t} = x_0 + \int_0^t e^{-[X]_s/2} dX_s$. $[\widehat{X}]_t = \int_0^{[X]_t} e^{-t} dt \leq 1$. And let $\hat{\xi}_s^n = e^{-[X]_t/2} \xi_s^n$)

Apply Kunita-Watanabe decompose:

$H_t \triangleq E^*(H \mid \mathcal{F}_t) = E^*(H) + \int_0^t \xi^H dX + N_t$.

if $\|N_T\|_\infty > 0$. Set $\frac{d\widehat{P}}{dP^*}\Big|_{\mathcal{F}_T} = 1 + (N_T / \|N_T\|_\infty)$

$\Rightarrow X$ is still $\widehat{P}$-mart ($X \perp N$). So: $\widehat{P} = P^*$

$\Rightarrow E^*(N_T) = \widetilde{E}(N_T)$. So: $N_T = 0$. a.s.

ii) $\Rightarrow$ i) Let $H = I_A = V^{I_A} + \xi \cdot X$. $A \in \mathcal{F}_t$.

$V^{I_A} = E^*(V_0) = E^*(V_T) = P^*(A)$ uniquely.

Cor. (Attainable Claims)

X is unit. satisfies NA on $(\Omega, \mathcal{F}_t, P)$

For $\forall$ bdd claim $H \in \mathcal{F}_T$

i) H is attainable.

ii) $E^*(H)$ doesn't depend on choice of EMM $P^* \sim P$ for X.

We have: i) $\Longleftrightarrow$ ii).

Pf: i) $\Rightarrow$ ii) By charac. of $X_t^H = \mathbb{E}^*(H|\mathcal{F}_t)$

we proved before c invar. of cheapest price)

ii) $\Rightarrow$ i) By KW decompose on H:

$\mathbb{E}^*(H) = \tilde{\mathbb{E}}(H)$

$= \mathbb{E}^*(H) + \tilde{\mathbb{E}}(\int_0^T \xi \, dX) + \tilde{\mathbb{E}}(N_T)$.

$= \mathbb{E}^*(H) + \mathbb{E}^*(\int_0^T \xi_s \, dX_s) + \mathbb{E}^*(N_T)$.

where $d\tilde{\mathbb{P}}/d\mathbb{P}^*|_{\mathcal{F}_T} = 1 + cN_T/\|N_T\|_\infty$

and X is still $\tilde{\mathbb{P}}$ - mart as before

$\Rightarrow \tilde{\mathbb{E}}(\int_0^T \xi_s \, dX_s) = 0$. $\mathbb{E}^*(N_T) = 0$.

$\tilde{\mathbb{E}}(N_T) = \mathbb{E}^*(N_T) + c\mathbb{E}^*(N_T^2)/\|N_T\|_\infty = 0$

So: $N_T = 0$. a.s.

$|\int_0^t \xi_s \, dX_s| \leq \mathbb{E}(|H| \,|\, \mathcal{F}_t) \leq \|H\|_\infty < \infty$.