Financial Modeling

(1) Brownian Financial Model:

① Structure:

Assume $(\mathcal{F}_t^B) := \sigma(B_r^i,\ 1 \le i \le d,\ 0 \le r \le t)$.

where $B = (B^1 \cdots B^d)$ is d-dim BM.

Let $r_t \in \mathcal{F}_t^B$-adapted, $L^1_{loc}(dt)$, $dS_t^0/S_t^0 = r_t\, dt$

Stock price $S_t \in \mathcal{F}_t^B$-adapted, right-conti.

<u>Thm:</u> (S_t) exhibits NFLVR on $(\Omega, \mathcal{F}, (\mathcal{F}_t^B),$
$\mathbb{P})$, has dynamics of form:

$$dS_t/S_t \overset{i)}{=} (r_t + \theta_t \cdot \sigma_t^B)\, dt + \sigma_t^B \cdot dB_t$$

$$\overset{ii)}{=} \mu_t\, dt + \sigma_t W_t, \quad t \ge 0.$$

for some pred. vector $\theta, \sigma^B \in L(W)$ and some 1-dim BM (W_t), $\sigma \in L(W)$, and pred. $\mu \in L^1_{loc}(dt)$, a.s.

<u>Pf:</u> $\mathbb{P}^* \sim \mathbb{P}$, local mart. p.m. for $X_t = e^{-\int_0^t r_s ds} S_t$. $Z_t = \frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t^B}$ is

Brownian $\mathbb{P}$-mart. $\Rightarrow \exists L = \int_0^\cdot Z_s / dZ_s$

Brownian $\mathbb{P}$-mart. s.t. $Z_t = \mathcal{E}(L)_t$.

By Repre. $\exists\, \theta \in L(B)$. $L_t = -\int_0^t \theta_s \cdot dB_s$.

XZ is also $\mathbb{P}$-local mart. As above:

$XZ. = \mathcal{E}(M).x_0$. $M = \int \lambda \cdot dB$. $\lambda \in L(B)$.

$$\Rightarrow S_t = e^{\int_0^t r_s ds} X_t Z_t / Z_t$$

$$= S_0 \exp\left(\int_0^t (\lambda_s + \theta_s) \cdot dB_s\right) \cdot \exp\left(\int_0^t r_s - \frac{1}{2}|\lambda_s|^2 + \frac{1}{2}|\theta_s|^2 + \frac{1}{2}|\lambda_s + \theta_s|^2 ds\right)$$

So: $\frac{dS_t}{S_t} = (\lambda_t + \theta_t) \cdot dB_t + (r_t + \theta_t \cdot (\theta_t + \lambda_t)) dt$.

Set $\sigma_t^B = \lambda_t + \theta_t$. We have i).

Let $W_t = \int_0^t \sigma_t^B / |\sigma_t^B| \, I_{\{|\sigma_t^B| \neq 0\}} + I_{\{|\sigma_t^B| = 0\}} dB_t$

Note $\langle W \rangle_t = t$. W is $\mathbb{P}$-local mart.

$\Rightarrow W_t$ is 1-dim BM. We have ii)

<u>Rmk</u>: i) Equation ii) means that we can compress the info. of d BMs into one BM $(W_t)_{t \geq 0}$.

ii) (θ_t^i) is market price of risk that $\mathbb{P}^*$ attributes to exposure to the risk dB_t^i.

iii) σ_t^B is intensity that S is driven by shock dB_t.

iv) $\mu_t - r_t = \theta_t \cdot \sigma_t^B$ is risk premium for holding S under $\mathbb{P}^*$.

<u>Cor.</u> $d=1$. & $\sigma_t^2 = \frac{d}{dt}[S]_t / S_t^2 > 0$. $\mathbb{P}\otimes dt$. a.e.

$\Rightarrow$ The Brownian financial market is complete. If $d \geq 2$. then it's incomplete.

<u>Rmk:</u> More generally. for m assets S_t^k:

i) NA $\Leftrightarrow$ $\mu_t = r_t + \theta_t \cdot \sigma_t$ has a solution θ_t. st. $\mathcal{E}(\int_0^\cdot \theta_t \, dB_t)$ $= Z_t$ is true mart. $\left(\sigma = \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^m \end{pmatrix}\right)$

ii) Complete $\Leftrightarrow$ d (number of risk factors B^i) $\leq \tilde{m}$ ("true" number of trade assets). i.e. σ has left inverse.

So: θ_t has only one solution.

($\Leftrightarrow$ Only exists unique risk-neutral p.m. $\mathbb{P}^*$ for $\bar{S}_t$)

<u>Pf</u>: Prove $\mathbb{P}^*$ is unique. equi. l.m.p.m.

$d\mathbb{P}^*/d\mathbb{P}|_{\mathcal{F}_t} = \mathcal{E}(-\int_0^{\cdot} \theta_s dB_s)$ by above

And we know $\theta_t = (\mu_t - r_t)/\sigma_t$

is the only choice. when $k = 1$.

(*): $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \sum x_k y_k$.

If $k > 1$. then $\theta_t \cdot \sigma_t^{\rho} \overset{(*)}{=} \mu_t - r_t$.

has more than one solutions.

<u>Rmk</u>: We can't directly use the Repre Thm: $\mathbb{E}^*(M \mid \mathcal{F}_T^B) = \mathbb{E}^*(M) + \int_0^t \theta_s dW_s^* = \mathbb{E}^*(M) + \int_0^t \frac{\theta_s}{\sigma_s X_s} dX_s$

as in const BS model.

Since $W_t^* = B_t + \int_0^t \frac{\mu_s(\omega) - r_s(\omega)}{\sigma_s(\omega)} ds$

may $\notin \mathcal{F}_t^B$ any more!

② <u>Robustness</u>:

Fix $\mathbb{P}^*$. EMM. and NA Brownian financial model have form $dS_t/S_t = r_t dt + \sigma_t dW_t^*$. W_t^* is $\mathbb{P}^*$ - BM. which can be reduced to specify:

a) interest rate r_t b) Volatility σ_t.

If $r \equiv 0$ (focus on volatility). σ unkown. and introduce a model under $\hat{\mathbb{P}}^*$:

A trader operates on this model and have estimate $(\hat{S}_t, \hat{\sigma}_t, \hat{W}_t^*)$. $\hat{\sigma}_t$ is const

i.e. have $d\hat{S}_t/\hat{S}_t = \hat{\sigma}\, d\hat{W}_t^*$. $\mathcal{F}_t = \mathcal{F}_t^{\hat{S}}$.

$\Rightarrow$ we want to price derivative $H = f(S_T)$ as $\hat{H} = f(\hat{S}_T)$. $\hat{H}_t(\omega) \overset{\text{M.p.}}{=} \hat{E}^*(f(\hat{S}_T) \mid \hat{S}_t = S_t(\omega))$ $= \hat{v}(t, S_t(\omega))$. Solve $\hat{v}(t,s)$ by BS - PDE:

$$\partial_t \hat{v}(t,s) + \frac{1}{2}\hat{\sigma}^2 s^2 \partial_s^2 \hat{v}(t,s) = 0. \quad \hat{v}(T,s) = f(s).$$

$\Rightarrow$ We want to use $H_0 = \hat{v}(0, S_0)$ and hedge strategy $\Delta_t = \partial_s \hat{v}(t, S_t)$.

Thm. If payoff func. f is convex. W satisfy $\sigma_t(\omega) \le \hat{\sigma}$. $\forall t \in [0,T]$. Then Δ_t superreplicate the claim $\eta = f(S_T)$. i.e.

$V_t = \hat{V}(0, S_0(\omega)) + \int_0^t \partial_S \hat{V}(u, S_u(\omega))\, dS_u(\omega) \geq$

$\hat{V}(t, S_t(\omega))$. and $V_T \geq f(S_T)$.

<u>Cor.</u> In the case $\sigma_t(\omega) \geq \hat{\sigma}$. We have Δ_t sub-replicate claim η.

<u>Pf:</u> Apply Itô's on $\hat{V}(t, S_t)$.

$$\hat{V}(t, S_t) = V_t + \frac{1}{2}\int_0^t \partial_S^2 \hat{V}(u, S_u)\, d[S]_u \quad = S_u^2 \sigma^2 du$$

$$+ \int_0^t \partial_t \hat{V}(u, S_u)\, du.$$

$$\overset{\text{BS-PDE on } V}{=} V_t + \frac{1}{2}\int_0^t (\sigma_u^2 - \hat{\sigma}^2) S_u^2 \partial_S^2 \hat{V}(u, S_u)\, du.$$

Next, we claim $\hat{V}(t, S)$ is convex:

$$\hat{V}(t, \lambda S^1 + (1-\lambda) S^2) = \hat{\mathbb{E}}(f(\hat{S}_T) \mid S_t = \lambda S^1 + (1-\lambda) S^2)$$

$$\overset{\text{mp.}}{=} \hat{\mathbb{E}}(f((\lambda S^1 + (1-\lambda) S^2)\, \mathcal{E}(\hat{\sigma}\hat{W})_{T-t}))$$

$$\overset{\text{convex}}{\leq} \lambda \hat{\mathbb{E}}(f(\cdots)) + (1-\lambda)\hat{\mathbb{E}}(f(\cdots))$$

$$\overset{\text{mp}}{=} \lambda \hat{V}(t, S^1) + (1-\lambda)\hat{V}(S^2).$$

So: $\partial_S^2 \hat{V} \geq 0$. $\hat{V}(t, S_t) \geq V_t$.

(2) <u>Stochastic Volatility model:</u>

Recall in BS model. Vega $:= \frac{\partial}{\partial \sigma}$ BS-call price $(T, k, S_0, r, \sigma) > 0$. So the price is strictly increasing. Given strike price k & maturity T. We observed a call price $\hat{V}_T$.

<u>Def</u>: Implied volatility $\sigma_{imp} = f^{-1}(\cdot, \vec{x})(C)$.

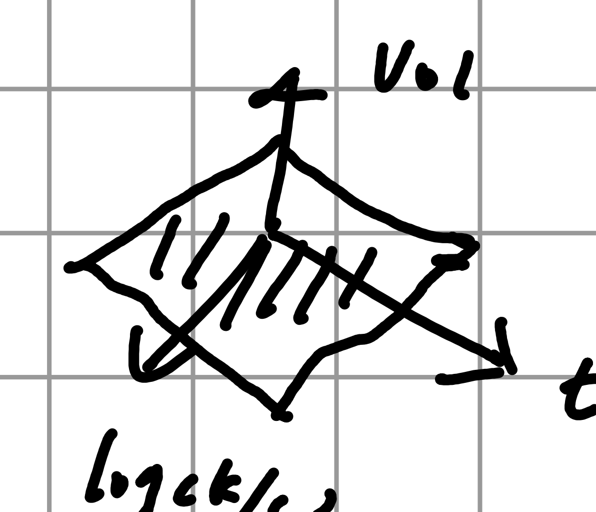
C is the real price of option and $f(\sigma, \vec{x})$ is the model price.

<u>e.g.</u> Let BS-call price (T, k, S_0, r, σ) = observed price

Apply IFT, we solve: $(T, k) \mapsto \sigma_{imp}(T, k, S_0, r)$.

<u>Rmk</u>: Given today's stock price S_0 and interest rate r.

$(T, k) \mapsto \sigma_{imp}(T, k, \sigma_0, r)$ is called volatility surface.



which accounts for the moneyness k/S_0 & maturity t.

better than using const. vol. before.

① Intro. of Vol. model:

i) <u>Hull & White</u>: $dS_t / S_t = \sigma_t \, dW_t^*$. W^* is P^*-BM.

where $d\sigma_t = \sigma_t(\alpha dt + \gamma dB_t)$. where B_t is BM correlated with W_t^*: $[B, W^*]_t = \rho^t \in [-1,1]$

Rmk: i) Note $\sigma_t = e^{\square}$. So it won't have zeros (i.e. always fluctuate).

ii) It has leverage effect. if $\rho^t < 0$. falling ~~stock~~ price ($W_t^* \downarrow$) go with high vol. ($B_t \uparrow$. when $\gamma > 0$)

iii) $\sigma_t = e^{\int^t \gamma dB_s + \int_0^t \alpha - \frac{\gamma^2}{2} ds}$

$$\xrightarrow{t\to\infty} \begin{cases} +\infty & \text{if } \alpha > \frac{\gamma^2}{2} \\ 0 & \text{if } \alpha < \frac{\gamma^2}{2} \end{cases}$$

iv) It's incomplete market since we have 2 risky terms (B, W^*). But only 1 asset X.

ii) Scott model: $dS_t / S_t = \sigma_t dW_t^*$.

where $\sigma_t^2 = e^{V_t}$. $dV_t = \alpha(\beta - V_t)dt + \gamma dB_t$.

B_t is another BM correlated with W_t^*.

Rmk: i) V_t is Ornstein-Uhlenbeck process so it's solvable, and it's also Gaussian.

ii) V_t has mean-reversion on β. i.e.

if $\alpha > 0$, $V \uparrow \Rightarrow \beta - V_t \downarrow \Rightarrow V \downarrow$.

So: V_t is stable.

iii) V_t allows stationary dist. ($\lim_{t \to \infty} V_t$ exist)

iv) For short term: ~~take~~ $\sigma_t = e^{V_0/2}$.

For long term. ~~take~~ $\sigma_t = e^{\beta/2}$.

v) It's incomplete.

iii) <u>Stein & Stein model</u>: $dS_t / S_t = \sigma_t \, dW_t^*$

where $d\sigma_t = \alpha(\beta - \sigma_t)dt + \gamma \, dB_t$. (O-U process). B_t is BM correlated with W_t^*.

<u>Rmk</u>: i) σ_t may ~~attain~~ zero.

ii) It also has prop. ii), iii), iv), v) as above.

iv) <u>Heston model</u>: $dS_t / S_t = \sigma_t \, dW_t^*$.

where $\sigma_t^2 = V_t$, $dV_t = \alpha(\beta - V_t)dt + \gamma V_t^{\frac{1}{2}} dB_t$

B_t is BM correlated with W_t^*.

<u>Rmk</u>: i) Yamada-Watanabe Thm. guarantees

the existence and uniqueness of V_t.

ii) V_t is affine process, which can be used

semi-analytic call-option price formula.

v) <u>Dupire's local model</u>: $dS_t/S_t = \sigma(t, S_t)\, dW_t^*$.

<u>Rmk</u>: i) It's nonparametric model. We need

to choose Σ_t on $\sigma(t, \cdot)$

ii) It's complete if $\sigma > 0$.

vi) <u>SABR</u> (Stochastic Alpha Beta Rho) Model:

$dS_t / S_t^{\beta} = \sigma_t\, dW_t^*$. $d\sigma_t = \alpha\, \sigma_t^{\beta}\, dB_t$. where

$[W^*, B]_t = \rho t$.

<u>Rmk</u>: i) It's popular in FX-trading.

ii) $\sigma_t \to 0$ $(t \to \infty)$

iii) $\beta = 1 \Rightarrow$ special case of Hull & White model.

iv) It's not complete.

② <u>On Dupire's local Vol model</u>:

σ is unknown, we can only use our model to obtain a estimate $\hat{\sigma}(t,S)$.

We first assume we've made the model calibration, i.e. let market price $H = f(S_T)$ equal model price $\hat{H} = f(\hat{S}_T)$ ($= \hat{\mathbb{E}}^*(f(\hat{S}_T))$ $\overset{\text{mark.}}{=} \hat{V}(0,S_0)$) for $\forall$ payoff func. f and $\forall$ maturity $T > 0$.

So: We also have $\mathbb{E}^*(f(S_T)) = \hat{\mathbb{E}}^*(f(\hat{S}_T))$

Next, we're going to find such $\hat{\sigma}$:

$$\hat{V}(0,S_0) = \hat{\mathbb{E}}^*(f(\hat{S}_T)) = \mathbb{E}^*(f(S_T)) = \mathbb{E}^*(\hat{V}(T,S_T))$$

$$\overset{\text{Itô}}{=} \hat{V}(0,S_0) + \mathbb{E}^*\left(\int_0^T \partial_t \hat{V}(t,S_t)dt + \int_0^T \partial_S \hat{V}(t,S_t)dS_t + \frac{1}{2}\int_0^T \partial_S^2 \hat{V}(t,S_t)d[S]_t\right)$$

By BS-PDE for $\hat{V}$, SDE for S_t. It's equi.:

$$0 \overset{\text{Fubini}}{=} \frac{1}{2}\int_0^T \mathbb{E}^*\left((\sigma_t^2 - \hat{\sigma}^2(t,S_t))S_t^2 \partial_S^2 \hat{V}(t,S_t)\right)dt$$

for $\forall T > 0$. So: it's equi.:

$$0 = \mathbb{E}^*\left((\mathbb{E}^*(\sigma_t^2 \mid S_t) - \hat{\sigma}^2(t,S_t))S_t^2 \partial_S^2 \hat{V}(t,S_t)\right)$$

We can choose $\hat{\sigma}^2(t,S_t) = \mathbb{E}^*(\sigma_t^2 \mid S_t)$.

Thm. (Gyöngy).

S_t has vol. σ under risk-neutral p.m. P^*.

If $\tilde{\sigma}(t,s) = \mathbb{E}^{*\frac{1}{2}}(\sigma_t^2(S_t) \mid S_t = s)$ is regular enough. Then: solution $\hat{S}$ of SDE:

$d\hat{S}_t / \hat{S}_t = \tilde{\sigma}(t, \hat{S}_t)\, d\hat{W}_t^*$ will satisfy

$Law(S_t \mid \mathbb{P}^*) = Law(\hat{S}_t \mid \hat{\mathbb{P}}^*)$.

Next, we want to determine $\hat{\sigma}^2(t,s)$ from market's data. First, we assume:

Call option price $C(T,K) = \mathbb{E}^*((S_T - K)^+)$ is observable for $\forall\, T, K$

1) Claim: (determine 1-dim marginal of S under $\mathbb{P}^*$ and $\hat{S}$ under $\hat{\mathbb{P}}^*$:

$\partial_K^+ C(T,K) = \mathbb{E}^*(\partial_K^+ (S_T - K)^+) = \mathbb{E}^*(-I_{\{K < S_T\}})$

$\Rightarrow \partial_K^2 C(T,K)$ determine density φ.

By Gyöngy Thm: We obtain density $\hat{\varphi}$ of $\hat{S}$ under $\hat{\mathbb{P}}^*$. ($\hat{\varphi} = \varphi$).

2) Note that density φ satisfies FPE:

$d\varphi(t,K)/dt = \frac{1}{2} \partial_K^2 (K^2 \hat{\sigma}^2(t,K) \varphi(t,K))$

From $C(t,K) = \int \varphi(t,s)(s-K)_+ ds$, we have:

$$\partial_t C(t,K) = \int \partial_t \varphi(t,S)(S-K)_+ dS$$

$$\overset{FPE}{=} \int \frac{1}{2}\partial_S^2\{S^2\sigma^2(t,S)\varphi(t,S)\}(S-K)_+ dS$$

$$\underset{\text{by part}}{\overset{\text{integrate}}{=}} \int \frac{1}{2} S^2 \sigma^2(t,S)\varphi(t,S)\delta_K(S)\, dS$$

$$= \frac{1}{2}K^2\sigma^2(t,K)\,\varphi(t,K).$$

(Note $\partial_K^2 (K-S)_+ = \partial_K I_{\{K>S\}} = \delta_S(K)$)

By i). we can replace σ, φ by $\hat{\sigma}$, $\hat{\varphi}$.

So: $\hat{\sigma}^2(t,K) = \frac{\partial_t C(\cdot\cdot)}{\hat{\varphi}(\cdot\cdot\cdot)} \cdot \frac{2}{K^2} \underset{\varphi=\hat{\varphi}}{\overset{i)}{=}} \frac{2}{K^2}\frac{\partial_t C(t,K)}{\partial_K^2 C(t,K)}$

i.e. We got Dupire's formula for $\hat{\sigma}$.

<u>Rmk</u>: i) We can only get finitely many option price $C(t,K)$ in reality, which's not enough to deri. $C(T,K)$

So we need interpolation schemes (e.g. assume $C(t,K)$ is polynomial)

ii) The interpolation should be NA.

Recall $\hat{C}(t,K)$ has NA $\Longleftrightarrow$

(a) $K \mapsto \hat{C}(t,K)$ $\searrow$ convex. And:

$\hat{C}(T,0+) = S_0$. $\lim_{K\uparrow\infty}\hat{C}(T,K) = 0$.

(b) $t \mapsto \hat{C}(t,K)$ $\nearrow$.

iii) The model lacks of consistency in time since the model is static (since we just apply the formula $\hat{\sigma}$ at $t=0$.) So it need conti. model recalibration.

⑧ <u>On Heston's vol. model:</u>

$d\hat{S}_t / \hat{S}_t = \sqrt{V_t}\, d\hat{W}_t^*$. $dV_t = \alpha(\beta - V_t)dt + \gamma\sqrt{V_t}\, dB_t$.

<u>Thm.</u> If $d \stackrel{\Delta}{=} 4\alpha\beta/\gamma^2 < 2$. We have $\hat{\mathbb{P}}^*(V_t = 0$. for some $t \geq 0) = 1$. If $d \geq 2$. $\hat{\mathbb{P}}^*(V_t = 0$. for some $t \geq 0) = 0$

<u>Rmk:</u> If $\alpha\beta$ is large enough, comparing to local vol. $\gamma^2 V_t$. Then level 0 won't be reached.

<u>Pf:</u> Apply Itô on $S(V_t)$ to find $S(\cdot)$. St. $S(V_t)$ is (u.i.) mart.

Itô's
$\Rightarrow \alpha(\beta - x)S'(x) + \frac{1}{2}\gamma^2 x S''(x) = 0$.

So: $S(x) = \int_0^x e^{2\alpha y/\gamma^2} y^{-d/2} dy$

we have $S'(x) > 0$. $S(0+) \begin{cases} = +\infty. & \frac{d}{2} \geq 1 \\ < +\infty. & \frac{d}{2} < 1. \end{cases}$

Recall $\mathbb{P}($M reaches a before $b) = \dfrac{b - M_0}{b - a}$. for mart M. and $0 < a < b$ by optional sampling Then if

$\tau_{a,b} =: \inf\{t \geq 0 \mid M_t \text{ hits } a \text{ or } b\} < \infty$.

Next, we prove: $\mathbb{E}^*(\tau_{a,b}) < \infty$:

$$\text{Since } +\infty > \overset{S \text{ bdd.}}{\ } \mathbb{E}^*(\lim_{t\to\infty} S^2(V_t^{\tau_{a,b}}))$$

Engineer $\longleftarrow$ perspective: See local mart. as mi. mart.

$$\overset{\text{Ito}}{\underset{\text{iso}}{=}} \mathbb{E}^*(\lim_{t\to\infty} \langle S(V_t^{\tau_{a,b}})\rangle_t)$$

$$\overset{\text{Ito}}{\underset{\text{form.}}{=}} \mathbb{E}^*(\int_0^{\tau_{a,b}} S'(V_s^{\tau_{a,b}})^2 \gamma^2 V_s^{\tau_{a,b}} ds)$$

$$\geq \mathbb{E}^*(\tau_{a,b}) \inf_{[a,b]} S'(x)^2 \gamma^2 x$$

So: $\mathbb{P}(V$ hits a before $b) =$

$\mathbb{P}(S(V)$ hits $S(a)$ before $S(b)) =$

$S(b) - S(V_0) \Big/ S(b) - S(a)$.

Let $a \downarrow 0$ and $b \to +\infty$. We have it.

<u>Thm.</u> If $d \geq 2$. $d \in \mathbb{Z}^+$. Then: $V_t \overset{d}{\sim} \sum_1^d (X_t^i)^2$

(X_t^i) is i.i.d Ornstein-Uhlbeck process

s.t. $dX_t^i = \frac{\gamma}{2} dW_t^i - \frac{\alpha}{2} X_t^i dt$.

Rmk: So $\sqrt{V}$ is radial part of d-dim O-U process. It also interprets why V_t never hits 0 if $d \geq 2$.

Thm. (Distributional prop. of Heston)

Density of $\log(\hat{S}_t/S_0)$ and call-option prices $\hat{E}^*((\hat{S}_T - K)^+)$ are calculable.

③ **Recent Model:**

i) Rough Vol. model: $d\hat{S}_t/\hat{S}_t = \hat{\sigma}_t \, d\hat{W}_t^*$. where $\hat{\sigma}_t = e^{X_t}$. $dX_t = \alpha(\beta - X_t)dt + \gamma \, dB_t^H$.

Rmk: In this model, it has at-the-money vol. skew: $\psi(T) = \left| \frac{\partial}{\partial k}\Big|_{k=0} \sigma_{imp}(k,T) \right|$

$\sim O(1/T^{\frac{1}{2}-H})$ $(T \downarrow 0)$. $k = \log K/S_0$.

ii) Local Stochastic Vol. model: $d\hat{S}_t/\hat{S}_t = a_t \, \sigma(t, \hat{S}_t) \, d\hat{W}_t^*$.

Rmk: Set $\hat{\sigma}_{Dup}(t, \hat{S}_t) = \hat{E}^*(a_t^2 \, \hat{\sigma}^2(t, \hat{S}_t) \mid \hat{S}_t)$

$d\tilde{\hat{S}}_t / \tilde{\hat{S}}_t = \hat{\sigma}^2_{Dup}(t, \hat{S}_t) \, d\hat{W}_t^*$. $\overset{\text{Gyöngy}}{\Rightarrow}$ $\mathcal{L}(\hat{S}) = \mathcal{L}(\tilde{\hat{S}})$

(3) Term Structure Models:

① Bonds:

Firms can fund themselves not only by issuing shares of stocks but also issuing bonds.

States also get money from collecting taxes and issuing bonds.

Rmk: Difference of stocks and bonds:

stocks give ownership of company & exposure of loss and profit. But bonds specify the future payments / coupons which exposes to inflation & defaults.

Def: Zero bond $(P_t(T))_{t\in[0,T]}$ pays its holder one currency at its maturity $t=T$. i.e. $P_T(T)=1$. is a nonnegative sto-chastic price process.

Rmk: Comparing bond prices with diff. T.

doesn't make sense because of accumulation of risk over time.

① **Forward rate agreement (FRA):**

We want to secure at time t for investing S-bond safely until $T > S$

- At time t: A sell 1 S-bond to B, get $P_t(S)$ € and buy $\frac{P_t(S)}{P_t(T)}$ T-bond.
- At time S: B receive 1 €. and pass it to A by FRA. as investment.
- At time T: A get $\frac{P_t(S)}{P_t(T)}$ €. B receive $1 + \tilde{L}_t(S,T)$. €. where $\tilde{L}_t(S,T)$ is interest rate.

$\Rightarrow$ For NA. we require $1 + \tilde{L}_t(S,T) = \frac{P_t(S)}{P_t(T)}$

Def: Link to zero bonds price $(P_t(\cdot))$. is

$$L_t(S,T) := \tilde{L}_t(S,T) / (T-S).$$

Rmk: $T-S$ is used for normalization.

S, we have $L_t(S,T) = \left(\frac{P_t(S)}{P_t(T)} - 1\right) / (T-S)$.

i) Set $S = t$. we have simple spot rate on $[t,T]$: $L_t(T) = \left(1/P_t(T) - 1 \right) / (T-t)$.

ii) Consider $T-S = \Delta$ is small enough. Then:

$$\Rightarrow \frac{P_t(S)}{P_t(T)} = 1 + L_t(S,T)(T-S) \overset{\text{Taylor}}{\simeq} e^{L_t(S,T)\Delta}.$$

We can define $R_t(S,T)$ conti. compound forward rate so. $P_t(S)/P_t(T) = e^{R_t(S,T)(T-S)}$.

by such approxi on each $[S, S+\Delta]$ with Markov property.

③ Short Rate model: $t \quad S\ T$

Next, we consider the rate over infinitesimal period $[S, S+dS]$. for $S > t$.

Set $F_t(S) := \lim_{T \downarrow S} R_t(S,T) = -\partial_S \log P_t(S)$.

So: $P_t(T) = \exp\left(-\int_t^T F_t(s)\, ds\right)$

Let $S \downarrow t$. We get short term interest $r(t) = F_t(t) = -\partial_S|_{S=t} \log P_t(S)$. $t \quad t+dt$.

We want to model short rate $(r_t)_{t\geq 0}$ on $(\Omega, (\mathcal{G}_t), \mathbb{P}^*)$ and make $\mathbb{P}^*$ an EMM for $e^{-\int_0^t r(s)ds} P_t(T)$. by setting $P_t(T) := \mathbb{E}^*(e^{-\int_t^T r(s)ds} \cdot 1 \mid \mathcal{G}_t)$.

<u>e.g.</u> i) (Vasicek) $dr_t = \alpha(\beta - r_t)dt + \gamma dW_t^*$.

ii) (CIR) $dr_t = \alpha(\beta - r_t)dt + \gamma\sqrt{r_t}\, dW_t^*$.

iii) (Dothan) $dr_t = \alpha r_t dt + \gamma r_t dW_t^*$.

<u>Rmk</u>: For more reflexible purpose, we can set α, β, γ to be $\alpha_t, \beta_t, \gamma_t$. (deterministic).

<u>e.g.</u> (Calibration for Ho-Lee model).

$dr_t = \alpha_t dt + \gamma dW_t^*$. $\Rightarrow r_t = r_s + \int \square$

$\mathcal{J}_s : \int_s^T r_t dt \sim N(m(s,T), v(s,T))$.

$m(s,T) = \int_s^T r_s + \int_s^t \alpha_u du\, dt = r_s(T-s) + \int \square$

$v(s,T) = var(\int_s^T \gamma(W_T^* - W_s^*) ds)$

$\overset{\text{Fubini}}{=} \gamma^2 (T-s)^3/3$.

$\Rightarrow \tilde{P}_s(T) = \mathbb{E}^*(e^{-\int_s^T r_t dt} \mid \mathcal{G}_s) = e^{-m(s,T) + \frac{1}{2}v(s,T)}$.

To Cali.: set $e^{-m(s,T)+\frac{1}{2}v(s,T)} =$

$e^{-\int_s^T F(u)du}$ (from the markets).

$\Rightarrow \partial_T F(T) = \alpha_T - \gamma^2 T$. We can choose

$\alpha(t,r)$ for calibration.

If $(r_t)_{t\geq 0}$ is Markovian (e.g. Ho-Lee, Vasicek..)

Then: $P_t(T) = \mathbb{E}^*(e^{-\int_t^T r_s ds} \mid \mathcal{F}_t) = p(t, r_t, T)$.

Def: The model for (r_t) is affine model if

$p(t,r,T) = e^{a(t,T) - b(t,T)r}$.

Thm. $dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t^*$. where

$\mu(t,r) = \alpha(t)r + \beta(t)$. $\sigma(t,r) = (\gamma(t)r + \delta(t))^{\frac{1}{2}}$

$\Rightarrow$ The short rate model is affine.

Pf. Note that the discounted price

$e^{-\int_0^t r_s ds} p(t, r_t, T)$ is a $\mathbb{P}^*$-mart.

Next, we assume $p(t,r,T) = e^{a(t,T)-\square}$

and find a. b. st.

$d(e^{-\int_0^t r_s ds} p(t, r_t, T))$ has zero dt-term:

$\partial_t a(t,T) - b(t,T)\beta(t) + \frac{1}{2} b(t,T)^2 \delta(t)$

$+ r_t(-1 - \partial_t b(t,T) - \alpha(t) b(t,T) + \frac{1}{2} b(t,T)^2 \gamma(t)) = 0 \quad (*)$

with boundary cond.:

$1 = P(T, r_t, T)$. i.e. $a(T,T) = b(T,T) = 0$.

We can solve $a(t,T)$, $b(t,T)$.

$(*) = A + r_t \cdot B = 0$, ~~for~~ $A = B = 0$, B is Riccati ODE

eg. (Vasicek model)

$\alpha(t) = -\bar{\kappa}$, $\beta(t) = \bar{\kappa}\bar{\theta}(t)$, $\gamma(t) = 0$, $\delta(t) = \bar{\gamma}$.

$\Rightarrow -1 - \partial_t b(t,T) + \bar{\kappa} b(t,T) = 0$. $b(t) = \frac{1}{\bar{\kappa}}(1 - e^{-\bar{\kappa}(T-t)})$

$a(t,T) = \int_t^T (b(t,T)\beta(t) - \frac{1}{2} \cdots)$

$F_0(T) = -\partial_T \log P(0, r_t, T) = e^{-\bar{\kappa}T}(r_0 + \cdots) - \cdots$.

④ Forward Rate model:

Next, we consider the model $P_t(T) = e^{-\int_t^T F_t(s)ds}$. It's easy to be calibrate from market data $F_t(s)$.

Thm. (HJM drift cond. for NA.)

If $dF_t(T) = \alpha_t(T)dt + \sigma_t(T) \cdot dW_t$. for some d-dim BM W_t. Then, the discounted price $e^{-\int_0^t r_s ds} P_t(T)$ satisfies NA $\Leftrightarrow$ $\exists$ d-dim process λ_t indept of $\forall T>0$, satisfies:

$$\alpha_t(T) = \sigma_t(T) \int_t^T \sigma_t(s) ds - \lambda_t \cdot \sigma_t(T).$$

<u>Pf</u>: $\log 1/P_t(T) = \int_t^T F_t(u) du$

$$\overset{\text{HJM}}{=} \int_t^T \{ F_0(u) + \int_0^t \alpha_s(u) ds + \int_0^t \sigma_s(u) dW_s \} du$$

$$\overset{\text{Fubini}}{=} \int_t^T F_0(u) du + \int_0^t \int_t^T \square$$

$$= \square + \int_0^t \int_s^T \square - \int_0^t \int_s^t \square$$

$$\overset{\text{Fubini}}{=} \int_0^T F_0(u) du + \int_0^t \int_s^T \square - \int_0^t \underbrace{F_u(u)}_{=r(u)}$$

$$\Rightarrow P_t(T) = e^{-\log 1/P_t(T)}$$

$$= P_0(T) - \int_0^t P_s(T) d(\log 1/P_t(T)) + \frac{1}{2} \int_0^t P_s(T) d[\log 1/P_t(T)]_s$$

So $dP_t(T) = -P_t(T) \{ (\int_t^T \alpha_t(u) du) dt + (\int_t^T \sigma_t(u) du) dW_t - r_t dt - \frac{1}{2} |\int_t^T \sigma_t(u)|^2 dt \}$

$$\Rightarrow d(e^{-\int_0^t r_s ds} P_t(T)) \overset{\Delta}{=} d\tilde{P}_t(T)$$

$$= \tilde{P}_t(T)\Big[\big(-\int_t^T \alpha_t(u)\,du + \frac{1}{2}\Big|\int_t^T \sigma_t(u)\,du\Big|^2\big)dt$$

$$- \big(\int_t^T \sigma_t(u)\,du\big)\,dW_t\Big].$$

To find Emm $\mathbb{P}^\lambda$. See $\frac{d\mathbb{P}^\lambda}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \mathcal{E}(\lambda)_T$

(λ should be indept of $T > 0$!)

Since $d\tilde{P}_t(T) = \tilde{P}_t(T)\Big[\big(-\int_t^T \alpha_t(u)\,du$

$$+ \frac{1}{2}\Big|\int_t^T \sigma_t(u)\,du\Big|^2 - \lambda_t\int_t^T \sigma_t(u)\,du\big)dt$$

$$- \int_t^T \sigma_t(u)\,du\,(dW_t - \lambda_t\,dt)\Big]$$

$= dW_t^\lambda$. $\mathbb{P}^\lambda$-BM

So: We require:

$$-\int_t^T \alpha_t(u)\,du + \frac{1}{2}\Big|\int_t^T \sigma_t(u)\,du\Big|^2 - \int_t^T \sigma_t(u)\,du \cdot \lambda_t = 0$$

Differentiate T at both sides.

<u>Thm:</u> (HJM drift cond. for $\mathbb{P}^*$-mart.)

If $dF_t(T) = \alpha_t^*(T)\,dt + \sigma_t(T)\,dW_t^*$

for some d-dim $\mathbb{P}^*$-BM W_t^*. Then:

$P_t(T)e^{-\int_0^t r_s}$ is $\mathbb{P}^*$-mart $\iff \alpha_t^*(T) = \sigma_t(T) \cdot \int_t^T \sigma_t(s)$

<u>Rmk:</u> We see drift α_t^* is totally determined by volatility $\sigma_t(T)$.

Pf: From equation of $d(\tilde{P}_t(T))$ above:

$$-\int_t^T \alpha_t^*(u)\,du + \frac{1}{2}\left|\int_t^T \sigma_t(u)\,du\right|^2 = 0$$

③ Numéraire & Forward:

- Note we always discount prices by evolution $e^{-\int_0^t r_s ds}$. Which is to compare future asset prices with today's price simultaneously.
- To generalize it, we can replace it by any strict positive process N_t, which is called numéraire. (e.g. $N_t = P_t(T)$, if NA)
- Assume $N > 0$ is price of some tradable asset and under P^*, $e^{-\int_0^t r_s ds} N_t$ is a P^*-mart. We define:

$$\frac{dP^N}{dP^*}\Big|_{\mathcal{F}_t} := e^{-\int_0^t r_s ds} N_t / N_0 \text{ gives p.m. } P^N.$$

$\Rightarrow$ if $C \geq 0$ is T-payoff, to price it:

$$\pi_t(C) := E^*(e^{-\int_t^T r_s ds} C \mid \mathcal{F}_t)$$

$$= E^*\left(C \cdot \frac{dP^N}{dP^*}\Big|_{\mathcal{F}_T} \cdot \frac{N_t}{N_T} \,\Big|\, \mathcal{F}_t\right) = E^{P^N}(C/N_T \mid \mathcal{F}_t) N_t$$

Set $N_t = P_t(T)$. $\frac{d\mathbb{P}^T}{d\mathbb{P}^*}\Big|_{\mathcal{G}_T} = e^{-\int_0^T r_s ds} / P_0(T)$

$\Rightarrow Z_t(C) = P_t(T)\, \mathbb{E}^{\mathbb{P}^T}(C \mid \mathcal{G}_t)$.

i.e. we can see the discount vanishes!

<u>Def</u>: i) Forward price Fwd_t^T is the price we agreed at time t. Under this contract, we pay $Fwd_t^T(C)$ & get payoff C at time T. without any other payment in (t, T).

ii) $\mathbb{P}^T$ is called forward measure for maturity T

<u>Rmk</u>: T-payoff are $\mathbb{P}^T$-mart. but not $\mathbb{P}^{T'}$-mart. if $T' \neq T$.

<u>Thm</u>: For T-payoff $C \geq 0$. $\mathbb{E}^{\mathbb{P}^T}(C \mid \mathcal{G}_t) = \frac{Z_t(C)}{P_t(T)}$ is the forward price for C contracted at time $t \leq T$.

<u>Pf</u>: Cash flow only happens at time T: $C - Fwd_t^T(C)$. For NA: price Z_t

$0 = Z_t(C - Fwd_t^T) = P_t(T)\, \mathbb{E}^T(C - Fwd_t^T \mid \mathcal{G}_t)$

i.e. we have $E^T(c \mid \mathcal{F}_t) = \text{Fwd}_t^T$.

<u>prop.</u> $F_t(T) = \mathbb{E}^{\mathbb{P}^T}(r_T \mid \mathcal{F}_t)$. $\forall t \in [0,T]$.

<u>Rmk:</u> So forward rates are best predictor for future short rates.

<u>Pf:</u> $F_t(T) = -\partial_T \log P_t(T) = -\partial_T P_t(T) / P_t(T)$

$$= -\partial_T E^*(e^{-\int_t^T r_s ds} \mid \mathcal{F}_t) / P_t(T)$$

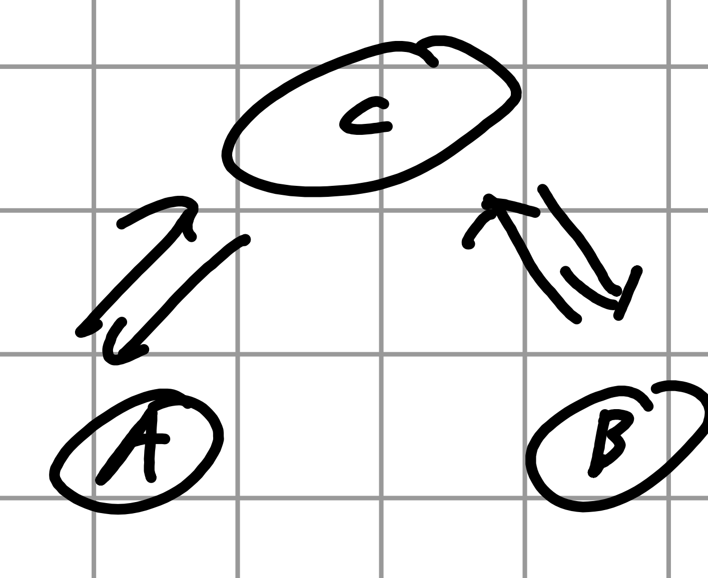
$$= \mathbb{E}^*(r_T \cdot \square \mid \mathcal{F}_t) / P_t(T).$$

$$= \pi_t(r_T) / P_t(T) = \mathbb{E}^{\mathbb{P}^T}(r_T \mid \mathcal{F}_t)$$

⑥ <u>Futures:</u>

Note that cash flows only happens at the maturity in forward contract, it's more like a bet, exposed to counterparty risk (since one may default.) So it only trades "over the counter" (OTC). where both know each others. (like a private contract) And it's not very liquid.

When it comes to futures, which can erase counterpart risk through a intermediary C. (clearing house) that asks A, B to post margins determined by futures price conti. on $[t,T]$.



Rmk: Futures have standard contract & it's more liquid.

Def: For T-payoff C, futures price $(Fut_t^T(C))_{t\le T}$ is determined by requiring $Fut_T^T = C$.

Rmk: At time t, the holder of the futures should pay size $dFut_t^T(C)$ and no further payment.

Thm: (Future prices)

Under pricing measure $\mathbb{P}^*$, with the numéraire $e^{\int_0^t r_s ds}$. If the $(Fut_t^T(C))$ has no arbitrary. Then: we have

$Fut_t^T(C) = \mathbb{E}^*(C \mid \mathcal{F}_t)$, $\forall t \in [0,T]$.

Pf: Consider strategy from $s \in [0,T]$.

St. holds $(\xi_t)_{[0,T]}$ futures for $c \in \mathcal{F}_T$

At time t, bank account β will evolve as follows:

$\beta_s = 0$. $d\beta_t = r_t \beta_t dt + \xi_t \, d\,Fut_t^T(c)$.

$=$ interest rate $+$ future part.

$$\Rightarrow \beta_t = e^{\int_s^t r_u du} \int_s^t e^{-\int_s^v r_u du} \xi_v \, dFut_v^T.$$

Let $\xi_v = 1 / e^{-\int_s^v r_u du}$.

Then: $\beta_T = Fut_{s,T}^T \exp(\int_s^T r_u du)$ is what we obtain at time T.

Since it's NA. and we began at zero.

$$\Rightarrow 0 = \pi_s(\beta_T) = \mathbb{E}^*(c|\mathcal{F}_s) - Fut_s^T(c).$$

Cor. If $r(t)$ is deterministic. Then:

$$Fut_t^T(c) = Fwd_t^T(c).$$

Pf: $p_t(T) = \mathbb{E}(e^{-\int_t^T r_s} | \mathcal{F}_t)$

$= e^{-\int_t^T r_s}$.

$\Rightarrow LHS = \mathbb{E}^*(e^{-\int_t^T r_s} c | \mathcal{F}_t) / p_t(T)$

$= \mathbb{E}^T(c | \mathcal{F}_t) = RHS.$