Finan. Optimization

(1) Merton's Optimiza:

- Consider a market with const. interest r. and stock price $S_t = S_0 e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}$.
- We want to maximize the expected termi. wealth $\mathbb{E}(V_T^{x,\theta})$. where $V_T^{x,\theta} := x + \int_0^T \theta_s dX_s$

$X_t := e^{-rt} S_t$ discounted price. But:

$\sup\limits_{\theta \text{ ad.}} \mathbb{E}(V_T^{x,\theta}) = +\infty$.

Pf: Set β is const. proportion strategy.

i.e. $dV_t = \beta V_t dX_t$

$\Rightarrow V_t = x \exp(\beta\sigma W_T + (\beta(\mu - r) - \frac{1}{2}\beta^2\sigma^2)T)$

So: $\mathbb{E}(V_T^{x,\theta}) = e^{\beta(\mu - r)T} \xrightarrow{\beta \to \infty} \infty$

where assume $\mu > r$ in engineer view (We can make more profit than put it in bank as expected).

$\beta > 1$ means we can borrow money.

unless $\mathbb{P}$ is mart. measure. But $\mathbb{E}(V_T^{x,\theta})$

$\equiv x$. which doesn't make sense!

Rmk: Note $V_T^{x,\theta} \to 0$ as $|\beta| \to \infty$. which

has contrary behavior to expect.

(as kind of compensation!). It

means high expected return is from

taking high risk.

Sol: We need to take risk aversion into

account! Merton introduce a method: to

maximize expected "utility" (which penalize

losses more than its reward gains.). i.e.

maximize $\mathbb{E}(u(V_T^{x,\theta}))$. $u(\cdot)$ is utility func.

eg. i) $u(x) = -e^{-qx}$. $q>0$ ii) $u(x) = \log x$.

① Dynamic program Prin.:

Idea: Don't focus on optimal strategy. But

on $u(T,x) = \sup_{\theta \text{ ad}} \mathbb{E}(U(V_T^{x,\theta}))$ value func.

Def: Value process $\mathcal{U}_t^\theta := u(T-t, V_t^{x,\theta})$

④ Mart. Optimize Prin.:

Intuition: i) Suboptimal strategy lead to worse value $\Rightarrow \mathcal{U}^\theta$ is supermart. for $\forall$ admissible θ

ii) Optimal strategy preserve value $\Rightarrow \mathcal{U}^{\theta^*}$ is mart for optimal θ^*.

Now suppose $u \in C^{1,2}$. Apply Itô's:

$$d\mathcal{U}_t^\theta = d u(T-t, V_t)$$
$$= (-\partial_t u(T-t,V_t) + \partial_x u(T-t,V_t)\theta_t X_t(\mu-r) + \frac{1}{2}\partial_x^2 u(T-t,V_t)\theta_t^2 X_t^2 \sigma^2)dt + \partial_x u(T-t,V_t) \cdot \theta_t X_t \sigma dW_t$$

where $V_t^{x,\theta} = x + \int_0^t \theta_s dX_s$.

With principle i). ii) above. We require:

$\square dt = 0$ for $\forall$ ad θ; $\square dt = 0$. $\forall \theta^*$.

i.e. We suggest that:

$$\sup_{\xi\in\mathbb{R}^1} \{ -\partial_t u(t,x) + \partial_x u(t,x)\xi(\mu-r) + \frac{1}{2}\partial_x^2 u(t,x) \cdot \xi^2\sigma^2\} = 0$$

$u(0,x) = U(x)$. called HJB equation.

If $\tilde{u}$ solve this HJB equation. Then:

$\mathbb{E}(U(V_T^\theta)) = \mathbb{E}(\tilde{u}(0, V_T^\theta)) = \mathbb{E}(\tilde{\mathcal{U}}_T^\theta)$

$\overset{\text{Supermart}}{\leq} \mathbb{E}(\tilde{\mathcal{U}}_0^\theta) = \tilde{u}(T, x).$

So performance of $\mathbb{E}(U(V_T^\theta))$ for any strategy θ can't be better than $\tilde{u}(T,x)$

If we let $\theta = \theta^*$. $V_T = V_T^*$. we have:

$\mathbb{E}(\tilde{\mathcal{U}}_T^{\theta^*}) = \mathbb{E}(\tilde{\mathcal{U}}_0^{\theta^*}) = \tilde{u}(T, x)$ by definition

i.e. $\mathbb{E}(U(V_T^*)) = \tilde{u}(T, X)$. attain max.

③ To derive HJB equation:

As we did in above:

i) identify state variables: maturity T & present wealth x in control system.

ii) Introduce value function $u(t,x)$ and value process $\mathcal{U}_t^\theta := u(T-t, V_t^\theta)$.

iii) Compute the semimart dynamics of value process from Itô's formula

iv) identify drift component and find conds on $u(t,x)$ ensure it's $\substack{\leq 0 \\ \geq 0}$ for $\substack{\max \\ \min}$ opt.

v) make sure the conds on $u(t,x)$ are the weakest for optimum the drift. i.e.

$$\sup_{\zeta \in \mathbb{R}^1} \{ -\partial_T u + \zeta(\mu - r)\partial_x u + \frac{1}{2}\zeta^2\sigma^2 \partial_x^2 u \} = 0$$

<u>Rmk</u>: i) The HJB equation is nonlinear PDE since the existence of $\sup_{\zeta}\{..\}$.

If ζ^* opt. attained : ($\partial/\partial\zeta^*$ both side)

$0 = (\mu - r)\partial_x u + \zeta^* \sigma^2 \partial_x^2 u$. So we solve

$\zeta^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{\partial_x u}{\partial_x^2 u}$. We obtain:

$$-\partial_T u - \frac{1}{2} \cdot \frac{\mu - r}{\sigma^2} \cdot \frac{(\partial_x u)^2}{\partial_x^2 u} = 0 \quad (HJB')$$

ii) If $\partial_x^2 u > 0$. we find $\zeta^* \to +\infty$ and $\sup_{\zeta}\{..\} \to +\infty$.

So ζ^* is maximizer $\Leftrightarrow \partial_x^2 u < 0$.

(If $\partial_x^2 u = 0$. Same case happens: It depend sign of $\partial_x u$)

④ <u>Verification Thm</u>:

Next, set $U(x) = \begin{cases} x^{1-q}/1-q, & x > 0 \\ -\infty, & x < 0 \end{cases}$ power utility

with $q > 0$. $q \neq 1$

Thm. The value function of Merton's problem for power utility is:

$$u(T,x) := \exp\left(\frac{1}{2}(1-\alpha)\frac{(\mu-r)^2}{\alpha\sigma^2}T\right)\frac{x^{1-\alpha}}{1-\alpha}, \quad x>0$$

and $X_t\theta_t^* = z^* V_t^*$, where $z^* = (\mu-r)/\alpha\sigma^2$. i.e. optimal strategy is always to invest the same fraction z^* of total wealth in stock.

Pf: 1°) Note value function satisfies:

$$u(T,\lambda x) = \sup_\theta \mathbb{E}(U(V_T^{\theta,\lambda x}))$$

$$= \lambda^{1-\alpha}\sup_\theta \mathbb{E}(U(V_T^{\theta/\lambda, x})) = \lambda^{1-\alpha}u(T,x)$$

So: $u(T,x) = x^{1-\alpha}u(T,1)$

$$\triangleq \frac{x^{1-\alpha}}{1-\alpha}f(T)$$

and $f(0)=1$. Since $u(0,x) = U(x)$

We put it inside HJB' equation

$\Rightarrow$ Obtain $f(T) = \exp\left(\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1-\alpha}{\alpha}T\right)$

So we also get $u(T,x) = \square$

And $\xi^* = -\frac{\mu-r}{\sigma^2}\frac{\partial_x u}{\partial_{xx} u} = \square$.

By def: $\theta_t^* = \zeta^*(V_t^*)/X_t = \frac{\mu - r}{\alpha \sigma^2} V_t^*/X_t$

ii) Next, we want to prove the value func. candidate we get in i) is the real value function. (denoted by $\hat{u}$)

First $\hat{u}(T,x) \geq \mathbb{E}(U(V_T^{\theta,x}))$ $\forall \theta$. ad.

WLOG, set $V_t^{\theta,x} \geq 0$. $\forall t$. otherwise:

$\mathbb{P}(\exists t>0.$ s.t. $V_t^{\theta,x} < 0) > 0 \overset{\text{NA}}{\Rightarrow} \mathbb{P}(V_T < 0) > 0$

So: $\mathbb{E}(U(V_T^{\theta,x})) = -\infty$. Suboptimal.

And By Itô's: $\hat{u}(T-t, V_t^{\theta,x}) = \hat{u}(T,x)$

$+ \int_0^t [-\partial_t \hat{u}(T-s, V_s) + \theta_t X_t (\mu - r) \partial_x \hat{u}(T-s, V_s)$

$+ \frac{1}{2}(\theta_t X_t \sigma)^2 \partial_x^2 \hat{u}(T-s, V_s)] ds$ Ⓐ

$+ \int_0^t \partial_x \hat{u}(T-s, V_s) \sigma \theta_t X_t dW_s$ Ⓑ

Since $\hat{u}$ solves HJB equation $\Rightarrow$ Ⓐ part (i.e. dt-part) ≤ 0

If $\alpha \in (0,1)$. Then Ⓑ part has lower bdd. So it's supermart.

$\Rightarrow \mathbb{E}(U(V_T)) = \mathbb{E}(\hat{u}(0, V_T)) \underset{\text{supermart}}{\overset{Ⓐ\leq 0}{\leq}} \mathbb{E}(\hat{u}(T, V_0)) = \hat{u}(T,x)$

To remove $q \in (0,1)$. we replace $\hat{u}(t,x)$ by $\hat{u}(t, x+\varepsilon)$ for some $\varepsilon > 0$ to avoid the explosion of $U(x)$ around $x=0$. So $\mathbb{E}(U(V_T+\varepsilon)) \le \hat{u}(T, x+\varepsilon)$

Let $\varepsilon \to 0$ and apply MCT.

Second, prove: $\exists \theta^*$. st. $\hat{u}(T,x) \le \mathbb{E}(U(V_T^*))$

Consider candidate optimal strategy with dynamics $V_0^* = x$. $dV_t^* = z^* V_t \, dX_t / X_t$

i.e. $V_t^* = \exp((z^*(\mu - r) - \frac{1}{2} z^{*2} \sigma^2) t + \sigma z^* W_t)$

Next, we prove $\hat{u}(T-t, V_t^*)$ is true mart. (Then $\mathbb{E}(U(V_T^*)) = \mathbb{E}(\hat{u}(0, V_T))$
$= \hat{u}(T,x)$)

With calculation above, we have Ⓐ-part $= 0$ after replace V_t^*.

So we only need to show Ⓑ-part is true mart. : By Itô's isometry,

Since $\mathbb{E}(\int_0^T (\partial_x \hat{u}(T-s, V_s^*, X_s \theta, \sigma)^2 ds)$

$= \int_0^T \mathbb{E}(\text{"some GBM"}) ds$ $\Rightarrow$ It's a

$= \int_0^T e^{c_2 s^2 + c_1 s + c_0} ds < \infty$. L^2-mart.

(2) Maximization via convex dual:

Consider financial model with 2 assets:

i) predictable interest r_t, s.t. $\int_0^t |r_s| < \infty$ $\forall t$

ii) conti. stock price S_t allowing NA.

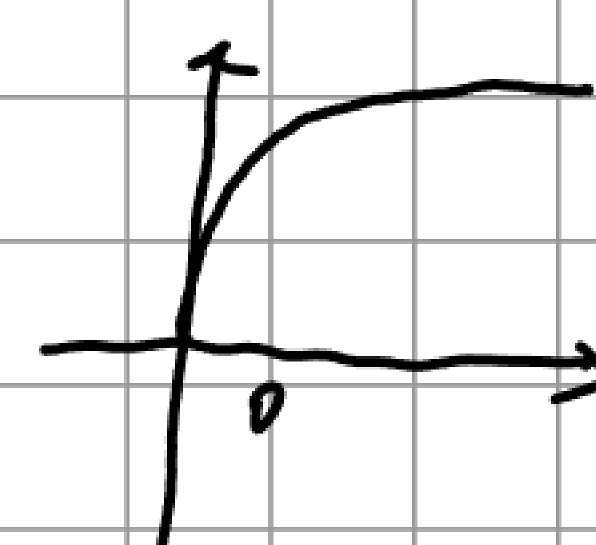
and with initial capital $x > 0$. Fix $(\Omega, \mathbb{P})$

Assume utility function $U(x)$ satisfies:

$U \in C^1$, $\uparrow$ strictly concave on $\mathbb{R}^{\geq 0}$ and

$U(x) = -\infty$ for $x < 0$. $U(0) = 0$ with

$U'(0+) = +\infty$, $U'(+\infty) = 0$ (Inada cond.)



Rmk: Recall in argument of HJB equ.:

$\partial_x^2 u < 0 \Rightarrow$ Maximizer exists. So intro. concave.

① Complete Case:

We assume the market is complete. i.e.

$\exists$ unique EMM $\mathbb{P}^*$ in the following.

Lemma. $H \geq 0 \in \mathcal{F}_T$ is dominated by $V_T^{x,\theta}$

discounted final wealth for some θ

ad. with $x > 0 \iff \mathbb{E}^*(H) \leq x$.

Rmk: So next, instead of opt. θ, we can maximize over $H > 0$.

Lemma. Assume $u(x) := \sup_{E^*(H) \le x} E(U(H)) < \infty$. Then $H^* \ge 0 \in \mathcal{F}_T$ is the terminal wealth of an optimal strategy with initial capital $x > 0 \iff E^*(H^*) = x$ and that: $E(U'(H^*)H) \le E(U'(H^*)H^*) < \infty$. for $\forall H \ge 0$ and $E^*(H) \le x$.

Rmk: It turns out H^* is also a optimizer for linear optimation.

Pf: $(\Leftarrow)$ By concave:

$U(H) - U(H^*) \le U'(H^*)(H - H^*)$

$\Rightarrow E(U(H)) - E(U(H^*)) \overset{\text{cond.}}{\le} 0.$

$(\Rightarrow)$ For opt. H^* and $H \ge 0$. $E^*(H) \le x$.

Set $H^\varepsilon := \varepsilon H + (1-\varepsilon)H^*$ still satisfies $E^*(H^\varepsilon) \le x$ and $H^\varepsilon \ge 0$

$0 \ge E((U(H^\varepsilon) - U(H^*))/\varepsilon)$ by Def.

And $RHS = \frac{U(H^\varepsilon) - U(H^*)}{H^\varepsilon - H^*} \cdot \frac{H^\varepsilon - H^*}{\varepsilon} I_{\{H^\varepsilon \neq H^*\}}$.

is monotone w.r.t $\varepsilon > 0$.

Set $\varepsilon \downarrow 0$. By MCT. we got that

$0 \geq \mathbb{E}(U'(H^*)(H - H^*))$.

with $U(0) - U(H^*) \leq U'(H^*)(-H^*)$

We have $\mathbb{E}(U'(H^*)H^*) < \infty$.

If $\mathbb{E}^*(H^*) < x$. We set $\tilde{H}^* = xH^* / \mathbb{E}^*(H^*) > H^*$. then by mono of U.

$\mathbb{E}(U(\tilde{H}^*)) > \mathbb{E}(U(H^*))$ Contradict!

<u>Cor.</u> $H^* \geq 0$. $\mathbb{E}^*(H^*) = x$. satisfies 1^{st} order cond.: $\mathbb{E}(U'(H^*)H) \leq \mathbb{E}(U'(H^*) H^*)$ for $\forall H \geq 0$. $\mathbb{E}^*(H) \leq x$ $\Leftrightarrow$ $\exists y > 0$. s.t. $H^* = (u')^{-1}(y \frac{dP^*}{dP})$ is the unique opt. where $y > 0$ is uniquely chosen to let $\mathbb{E}^*(H^*) = x$.

<u>Pf:</u> 1) Unique: if H^*. $\tilde{H}^*$ both opt.

Set $H = (H^* + \tilde{H}^*)/2$. then $\mathbb{E}^*(H) = x$ and by strictly concave of $U(x)$. we have:

$\mathbb{E}(U(M)) > \frac{1}{2}(U(x)+U(x)) = U(x)$

which's a contradiction!

2) Next, we prove: $\exists y > 0$, s.t. $H^y = (U')^{-1}(y\, dP^*/dP)$ has $\mathbb{E}(H^y) = x$.

Note $dP^*/dP > 0$. $y \mapsto \mathbb{E}^*(H^y)$ will be conti. (by Mono. Converge Thm) and monotone. $\mathbb{E}^*(H^y) \to \infty$ if $y \downarrow 0$ and $\mathbb{E}^*(H^y) \to 0$ if $y \to \infty$

So it's bijection from $\mathbb{R}^{\geq 0}$ to $\mathbb{R}^{\geq 0}$.

3) Next, we show H^y is optimal.

$\mathbb{E}(U'(H^*)H) = \mathbb{E}(yH\frac{dP^*}{dP}) = y\mathbb{E}^*(H)$

$\leq xy = \mathbb{E}^*(yH^*) = \mathbb{E}(U'(H^*)H^*)$.

e.g.) (Application in Black-Scholes market)

Consider $r > 0$. $dX_t/X_t = (\mu - r)dt + \sigma dW_t$ stock price dynamics. This is a complete market as we see before. with $dP^*/dP = \mathcal{E}(-\theta W_T)$. $\theta = \mu - r/\sigma$.

Let $U(x) = x^{1-\gamma}/1-\gamma$ power utility.

$$\Rightarrow M^{\eta} = y^{-\frac{1}{\gamma}} (dP^*/dP)^{-1/\gamma}$$

$$= y^{-\frac{1}{\gamma}} X_T^{\theta/\gamma\sigma} / (E(X_T^{-\theta/\sigma}))^{-\frac{1}{\gamma}}$$

$$= \text{const.} \cdot X_T^{p}. \quad p = \theta/\sigma\gamma.$$

i.e. the optimal strategy is to "all in" power option with $p = \theta/\sigma\gamma$ at time T. ($E^*(M^{\eta}) = x$ is initial capital)

<u>Rmk</u>: There's two ways to get the optimal above. One is "$\theta_t dX_t$" i.e. apply strategy θ_t continuously on stock for $\forall t \le T$.

Another is to apply derivative investment at time T as above! Push all money on derivative is more practical!

Recall to replicate the power option.

$$E^*(e^{-rT} X_T^p \mid \mathcal{F}_t) = X_t^p \exp((p-1)(\tfrac{1}{2}p\sigma^2 (T-t) + rT))$$ is the discounted price.

since it's a martingale. So:

$\xi_t \, dX_t = dX_t^p = pX_t^{p-1} \exp(\square) \, dX_t$

i.e. $\xi_t = p X_t^{p-1} \exp(\square)$. is repli.

$c\xi_t = p \, E^*((e^{-rT} X_T^p \mid \mathcal{F}_t)) / X_t$

$= p V_t^* / X_t$.

V_t^* is NA price for c many power option

$\Rightarrow c\xi_t X_t / V_t^* = p = \theta / \alpha\sigma = \dfrac{\mu - r}{\alpha \sigma^2}$.

Which coincides with the result of HJB equation in Merton's prob. before.

② Incomplete case:

Set $\mathcal{P} := \{$ equi. local mart. p.m. for $X \}$. $|\mathcal{P}| > 1$.

Thm (Optional decomposition)

right-conti. process $U \geq 0$ is $\mathcal{P}$-supermart. (i.e. $\forall P \in \mathcal{P}$. U is P-supermart.)

$\Leftrightarrow U_t = U_0 + \int_0^t \theta_s \, dX_s - A_t$. Where θ is predictable and dX-integrable and $A \geq 0$ $\nearrow$. right-conti and adapted.

Rmk: i) It's variant of Doob-Meyer's decompose. But note that we require A_t to be adapted & right conti. (i.e. optional). rather predictable. So this is why we call it optional Thm.

ii) The decomposition isn't unique.

Thm (Doob-Meyer's decompose)

$\forall$ local submart X can be uniquely decomposed as $X = M + A$. where $M \in \mathcal{M}^{loc}$. $A \uparrow$. predictable. right-conti.

Rmk: i) e.g. For $M \in \mathcal{M}_c^{loc}$. $\Rightarrow M^2$ is submart. then $A = \langle M \rangle$

ii) X can be only càdlàg (modif.)

Cor. $\forall$ local submart is semimart.

Pf: For simplicity. $\forall$ Mart follows are conti.

($\Leftarrow$) is trivial $E(U_T) \le E(U_0)$. $\forall T$

For ($\Rightarrow$): Fix $\mathbb{P}^0 \in \mathcal{P}$. Apply Doob-Meyer decomposition: $U_t = U_0 + M_t^0 - A_t^0$.

$M^0 \in \mathbb{P}^0 - \mathcal{M}_c^{loc}$. $A^0 \geq 0$. $\uparrow$. predictable.

Next. we show: $M_t^0 = \int_0^t \theta_s dX_s$ for some $\theta \in L(X)$.

$\Leftrightarrow$ M^0 is a $\mathcal{P}$-local martingale.

(Since $\mathbb{E}^{P^*}(M^0) = 0$ doesn't depend on $P^* \in \mathcal{P}$. $\xRightarrow{\text{localize}^{(*)}}$ it's attainable!)

$\Leftrightarrow$ $\langle M^0, L \rangle = 0$. for $\forall P^* \in \mathcal{P}$. where $\mathcal{E}(L)$ is density of $\frac{dP^*}{dP^0}$.

(*) $\tau_n = \inf\{t \geq 0 \mid |M_t^0| \geq n\} \to \infty$ since M^0 conti. $\Rightarrow$ local bdd

If $\exists$ some such L. s.t. $\langle M^0, L \rangle \neq 0$.

Set $\tau_n = \inf\{t \geq 0 \mid \mathcal{E}(L) \leq \frac{1}{n} \text{ or } \mathcal{E}(L) \geq n\}$

let $dP^*_{q,n}/dP^0 = \mathcal{E}(qL)_{T \wedge \tau_n}$

1) $P^*_{q,n} \in \mathcal{P}$. for $\forall |q| \geq 1$.

$\mathcal{E}(qL) = \mathcal{E}(L)^q e^{\frac{1}{2}(q-q^2)[L]}$ is bdd.

$\Rightarrow$ $\mathcal{E}(qL)$ is u.i. P^0-mart.

And $\langle X, qL^{\tau_n} \rangle = q \langle X, L \rangle^{\tau_n} = 0$.

So: $P^*_{q,n} \in \mathcal{P}$.

2) $U_t = U_0 + (M_t^0 - \langle M^0, qL^{\tau_n} \rangle_t) - (A_t^0 - \langle M^0, qL^{\tau_n} \rangle)$

Since U_t is $\mathbb{P}^*_{\alpha,n}$ - supermart. So:

$A_t^0 - \alpha \langle M^0 . L^{\tau_n} \rangle_t \geq 0$. $\uparrow$. $\forall |\alpha| \geq 1$. $\forall n$.

Note $\tau_n \uparrow \infty$. $(n \to \infty)$

$\mathbb{P}^0 (A_t^0 - \alpha \langle M^0 . L^{\tau_n} \rangle_t \geq 0) = 1 \leq \mathbb{P}^0 (\tau_n \leq t)$

$+ \mathbb{P}^0 (\tau_n > t , \square)$. Set $n \to \infty$. $T \to \infty$

and $n \to \infty$. $\alpha \to -\infty$. We have:

$\mathbb{P}^0 (\langle M^0 , L \rangle < 0) = \mathbb{P}^0 (\langle M^0 , L \rangle > 0) = 1$.

Which's a contradiction!

<u>Cor</u>. $|\mathcal{P}| > 1 \Rightarrow |\mathcal{P}| = +\infty$ (By i))

<u>Thm</u> (Superreplication)

The super-replication price process for H is $U_t := \operatorname{ess\,sup}_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* (H | \mathcal{F}_t) = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* (H)$

$+ \int_0^t \theta \, dX - A_t$. $\forall t \in [0,T]$. for some ad.

θ . $A_t \uparrow \geq 0$. optional process.

<u>Pf</u>: U_t is a $\mathcal{P}$ - supermart. $\Rightarrow$ Apply opt. decomp

<u>Cor</u>. $H \in \mathcal{F}_T \geq 0$. satisfies $H \leq x + \int_0^T \theta \, dX$

for some ad. $\theta \iff \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* (H) \leq x$.

Lemma. (Komlos's)

Seq of r.v.'s (X^n). s.t. $X_n \geq 0$. on $(\Omega, \mathcal{F}, \mathbb{P}) \Rightarrow \exists\, \tilde{X}^n \in \mathrm{conv}\{X^n, X^{n+1} \cdots\}$

s.t. $\tilde{X}^n \xrightarrow{a.s.} X$ r.v. take value $\in \overline{\mathbb{R}}^{\geq 0}$.

Pf: Set $U(x) = 1 - e^{-x}$. and consider

$u_n := \sup_{\tilde{X} \in \mathrm{conv}\{X^n, \cdots\}} \mathbb{E}(U(\tilde{X})) \downarrow$

So : $u_n \to \inf u_n =: u_\infty$

Let $\tilde{X}^n \in \mathrm{conv}.\{X^n, \cdots\}$. s.t. $\mathbb{E}(u(\tilde{X}^n)) \to u_\infty$.

prove: $(\tilde{X}^n)$ converges in pr.

By strictly concave & C^1 of $U(x)$

$\exists \beta > 0$. s.t. $u(\frac{1}{2}(x+y)) \geq \frac{1}{2}u(x) + \frac{1}{2}u(y) + \beta I\{x \vee y \leq N,\ |x-y| \geq \varepsilon\}$.

($f(x,y) = u(\frac{1}{2}(x+y)) - \square$ can't attain 0 on cpt set $\{|x| \leq N, |y| \leq N\}$.)

Set $x = \tilde{X}^n$, $y = \tilde{X}^m$. take $\mathbb{E}(\cdot)$.

LHS $\leq u_{m \wedge n}$. Let $m, n \to \infty$. So:

$\beta \lim_{m,n} \mathbb{P}(|\tilde{X}_n - \tilde{X}_m| \geq \varepsilon,\ \square \leq k) \leq 0$

S_0 : $\mathbb{P}(|\tilde{X}_n - \tilde{X}_m| \geq \varepsilon) \to 0$

$\Rightarrow$ $\tilde{X}_n \to X$ in pr.

δ_0 : $\exists\, \tilde{X}_{n_k} \in \mathrm{conv}\{X_k, \cdots\} \xrightarrow{a.s.} X$.

Thm. (Existence of optimazer)

Assume $u(x) = \sup_{\mathbb{E}^*(H) \leq x,\ \forall \mathbb{P}^* \in \mathcal{P}} \mathbb{E}(U(H)) < \infty$.

$\overline{\lim}_{y \to \infty} u(y)/y = 0$. (sublinear growth) $\Rightarrow$

$\forall x > 0$. $\exists H^* > 0$ $\mathbb{P}$-a.s. unique. st. $H^* =$

$\arg\max_{H \geq 0,\ \mathbb{E}^*(H) \leq x,\ \forall \mathbb{P}^* \in \mathcal{P}} \mathbb{E}(U(H))$.

Rmk: If U is bdd from above. then:

$\exists$ unique optimazer to the UMP.

Pf: 1) Unique: as before, by strictly concave

2) Take $H^n \geq 0$. st. $\mathbb{E}^*(H^n) \leq x$. for $\forall \mathbb{P}^* \in \mathcal{P}$. & $\mathbb{E}(U(H^n)) \to u(x)$.

By Komlos's Lemma: $\exists \tilde{H}^n \in \mathrm{conv}\{X^n, \cdots\}$. st. $\tilde{H}^n \xrightarrow{a.s} H^*$.

First, we note $\mathbb{E}^{\mathbb{P}}(\tilde{H}^n) \leq x$. $\forall n$.

$u(x) \geq \mathbb{E}(U(\tilde{H}^n))$

$\overset{\text{concave}}{\geq} \sum_{m \geq n} \lambda^n_m \mathbb{E}(U(H_m))$

$\geq \sum \lambda^n_m (u(x) - \varepsilon). = u(x) - \varepsilon$

for some $\varepsilon > 0$ and n large enough.

So: $\mathbb{E}(U(\hat{H}^n)) \to u(x)$. as well.

As for H^*: we check it's opt.:

$\mathbb{E}(H^*) \overset{\text{Fatou's}}{\leq} \underline{\lim} \, \mathbb{E}(\hat{H}^n) = x$. by $H^* \geq 0$

And $\mathbb{E}(U(H^*)) \overset{\text{u.i.}}{=} \lim \mathbb{E}(U(\hat{H}^n))$

$= u(x)$

u.i. follows from $u(x)/x \to 0$. $x \to \infty$:

<u>Pf</u>: (Wrong proof).

WLOG. $U(x) \xrightarrow{x\to\infty} \infty$. Or $U(\hat{H}^n)$ are u.i. ✓

$u(x) \geq U(x)$. by set $H = x$.

$\Rightarrow U(x)/x \to 0$. $(x \to \infty)$

Set $\varepsilon_m = \sup_{U(x) \geq m} U(x)/x \to 0. (m \to \infty)$

$\mathbb{E}(U(\tilde{H}^n) I_{\{U(\tilde{H}^n) \geq m\}}) \leq$

$\mathbb{E}(\varepsilon_m \tilde{H}^n I_{\{\cdots\}}) \leq \varepsilon_m x \xrightarrow{m\to\infty} 0$.

<u>Rmk</u>: It's wrong because $\mathbb{E}(H)$

may not $\leq X$ except $\mathbb{P} = \mathbb{P}^*$.

Then. in this case:

$$\mathbb{E}^*(U(H)) \overset{\text{Jensen}}{\leq} U(\mathbb{E}^*(H)) \overset{\text{mono}}{\leq} U(x).$$

i.e. $U(x) = u(x)$. trivial case!

<u>Lemma.</u> $(X_i)_I$ is u.i. $\Longleftrightarrow$ (X_i) is L^1-bdd and $\forall$ decompose $\Omega = \sum A_k$. We have: $\overline{\lim_{n\to\infty}} \sup_I \mathbb{E}(|X_i| I_{A_n}) = 0$.

Note $\mathbb{E}(u(\tilde{H}_n)) \to u(x)$. So it's L^1-bdd

By contradiction: $\exists (A_n)$. s.t. $\Omega = \sum A_n$. and $\mathbb{E}(U(\tilde{H}^n) I_{A_n}) \geq \varepsilon$

Set $R_n := \sum_1^n \tilde{H}^m I_{A_m} \geq 0$.

$$x_n \overset{\Delta}{=} \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*(R_n) \leq \sup \sum_1^n \mathbb{E}^*(\tilde{H}_m) \leq n x$$

$$u(x_n) \geq \mathbb{E}(U(R_n)) \overset{\text{concave}}{\geq} \sum_1^n \mathbb{E}(U(\tilde{H}^m) I_{A_m})$$

$$\geq n\varepsilon \qquad \text{So: } u(x_n)/x_n \geq \varepsilon.$$

But $x_n \geq n\varepsilon \to \infty$. if $x \to \infty$.

Contradiction with sub-linear growth.

<u>Thm.</u> (First order condition in incomplete).

$H^* \geq 0$ is terminal wealth of optimal strategy with initial $x \iff \sup_{\mathcal{P}} \mathbb{E}^*(H^*) = x$ and $\mathbb{E}(u'(H^*)H) \leq \mathbb{E}(u'(H^*)H^*) < \infty$.

for $\forall H \geq 0$ and $\sup_{\mathcal{P}} \mathbb{E}^*(H) \leq x$.

<u>Convex duality:</u>

$\forall H \geq 0$. $H \leq x + \int_0^T \theta \, dX$ for some admissible θ and $\forall \mathbb{P}^* \in \mathcal{P}$. we have:

$$\mathbb{E}(U(H)) \overset{a)}{\leq} \mathbb{E}(U(H)) - y(\overbrace{\mathbb{E}^*(H) - x}^{\leq 0})$$

$$= \mathbb{E}(U(H) - y\frac{d\mathbb{P}^*}{d\mathbb{P}}H) + xy.$$

Set $V(y) = \sup_{x > 0} \{U(x) - xy\}$

So: LHS $\overset{b)}{\leq} \mathbb{E}(V(y\frac{d\mathbb{P}^*}{d\mathbb{P}})) + xy$.

$$\Rightarrow \sup_{\substack{H \geq 0.\ \mathbb{E}^*(H) \leq x \\ \text{for } \forall \mathbb{P}^* \in \mathcal{P}}} \mathbb{E}(U(H)) \overset{c)}{\leq} \inf_{y > 0} \inf_{\mathbb{P}^* \in \mathcal{P}} \{\mathbb{E}(V(y\frac{d\mathbb{P}^*}{d\mathbb{P}})) + xy\}.$$

$$= \inf_{y > 0} \{\mathcal{V}(y) + xy\}.$$

<u>Rmk:</u> i) $V(y)$ is Legendre-Fenchel transf. of $U(x)$. $\Rightarrow V(y)$ is convex.

ii) If the "=" above can be attained for $\forall x, y, \mathbb{P}^*$ (fix), then:

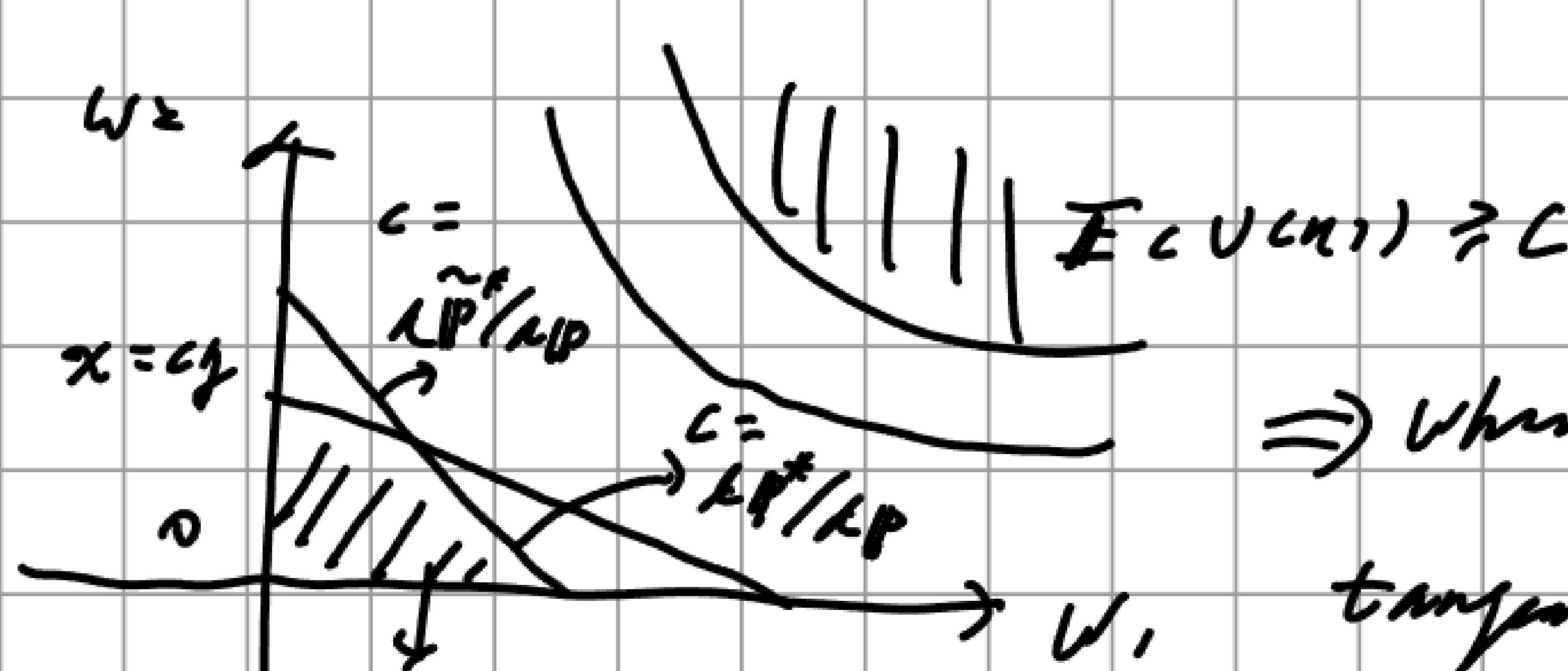
$$\begin{cases} \mathbb{E}^*(n^*) = x & \text{by a).} \\ U'(n^*) = y \dfrac{d\mathbb{P}^*}{d\mathbb{P}} & \text{by b).} \end{cases}$$

and from

c): $u(x) = \inf\limits_{y>0} \{V(y) + xy\}$. So that $u(x)$ and $V(y)$ are in duality with $V(y) = \inf\limits_{x>0} \{u(x) - xy\}$.

iii) The problem in ii) is that which $\mathbb{P}^* \in \mathcal{P}$ we should choose?

Visualization:



budget set $= \{n \geq 0,\ \mathbb{E}^*(n) \leq x,\ \forall \mathbb{P}^*\}$.

$\Rightarrow$ when $\mathbb{E}(U(n))$ tangent to the bdry of budget set $\Rightarrow$ optimal. i.e. $c = u(x)$

$$\Rightarrow U'(n^*) = y \frac{d\mathbb{P}^*}{d\mathbb{P}}$$