Continuous-Time Mathematical Finance

(a.k.a. Finanzmathematik II)

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This version: Berlin, November 25, 2024

Review of FiMa I and motivation of FiMa II

- Definition discrete-time financial model $\bar{S} = (S_t^0, S_t)_{t=0,1,\dots,T}$
- Wealth dynamics from selffinancing strategies
- Absence of arbitrage characterized by FTAP
- Arbitrage-free contingent claim prices
- Complete and incomplete markets
- (Super-)replication
- Primer on continuous-time Black-Scholes model: Ito isometry, Ito processes, Ito formula, Girsanov change of measure, ...

But:

- No really relevant financial models in FiMa I: essentially only binomial CRR-model
- Conceptual gaps in continuous-time Black-Scholes theory: completeness, merely mathematically convenient description of strategies, ...

Part I

Foundations of continuous-time financial models

Outline of Part I: Foundations of continuous-time financial models

Gains, losses and stochastic integration

Ito's formula: trend and volatility

No arbitrage and martingale measures

Pricing and Hedging

Outline

Outline

Continuous-time financial models

- $(\Omega, (\mathscr{F}_t), \mathbb{P})$ filtered probability space (stochastic basis)
- $\bar{S} = (S^0_t, S_t)_{t\in[0,T]}$ rightcontinuous stochastic process with left limits (càdlàg , RCLL) describing price evolution of numéraire $S^0 > 0$ and risky asset(s) S_t; adapted to
- $(\mathscr{F}_t)_{t\in[0,T]}$ filtration describing information flow
- $\bar{\xi} = (\xi^0_t, \xi_t)_{t\in[0,T]}$ predictable process describing investment strategy

? *What is "predictable" supposed to mean in continuous time?*

! There is no "next" or "previous" period any more!

? *How are we to interpret continuous-time trading in practical terms where positions can be changed rapidly, but not continually?*

? *How to describe profits and losses from continuous trading mathematically?*

Simple strategies

All these questions are very easy to answer for **simple strategies** of the form

$$\bar{\xi}_t = \sum_{i=1}^{n} \bar{\xi}_{T_i} 1_{(T_{i-1}, T_i]}(t), \quad t \in [0, T],$$

for finite stopping times

$$0 =: T_0 \leq T_1 \leq \cdots \leq T_n := T$$

specifying n periods $(T_{i-1}, T_i]$ over which the positions are held constant:

$$\bar{\xi}_{T_i} \in \mathscr{F}_{T_{i-1}} \rightsquigarrow \text{fixed at beginning of period, non-anticipative,}$$

namely **just as in discrete time!**

Remark

It will be convenient in our discussion to focus on a finite time horizon $T \in [0, \infty)$. Much of what we will have to say will be valid—mutatis mutandis—also for $T = \infty$ though.

PnL from selffinancing simple strategies

Selffinancing condition for simple strategies:

$$\bar{\xi}_{T_i}\bar{S}_{T_i} = \bar{\xi}_{T_{i+1}}\bar{S}_{T_i}, \quad i = 1, \ldots, n-1.$$

In discounted quantities

$\bar{X} := (X^0, X) := \bar{S}/S^0 = (1, S_t/S_t^0)_{t\in[0,T]}$ $\rightsquigarrow$ discounted asset prices

$V := V^{\bar{\xi}} := \bar{\xi}\bar{X}$ $\rightsquigarrow$ discounted wealth

this means that, with $v := \bar{\xi}_{T_1}\bar{S}_0$ denoting the initial wealth,

$$V_t = V_t^{v,\xi} := v + \underbrace{\sum_{i=1}^{n} \xi_{T_i}(X_{T_i\wedge t} - X_{T_{i-1}\wedge t})}_{=:G_t(\xi) \text{ gains from trading by time } t}, \quad t \in [0, T],$$

describes the discounted wealth evolution for a selffinancing and simple strategy $\bar{\xi}$.

? *How to extend the definition of V or G to continually changing $\bar{\xi}$?*

Consistent and stable gain process specifications

Want: Consistent and stable extension of

$$\xi \mapsto G(\xi)$$

to as large a class of strategies ξ as possible.

? *What do we mean by "consistent" and "stable"?*

- "consistent": restriction to simple strategies yields above specification
- "stable": continuity property in the sense that similar strategies $\xi \approx \xi'$ should yield similar gains $G(\xi) \approx G(\xi')$

? *Continuity in what sense?*

Mathematically natural (and to some extent also natural from a financial-economic perspective) continuity requirement:

$$\sup_{t \in [0,T]} |\xi_t^n - \xi_t| \to 0 \text{ uniformly, i.e., in } L^\infty(\mathbb{P})$$

$$\Rightarrow \quad G_t(\xi^n) \to G_t(\xi) \text{ in probability for each } t \in [0, T].$$

Bichteler-Dellacherie Theorem

Theorem

The continuity requirement

$$\sup_{t\in[0,T]} |\xi_t^n - \xi_t| \xrightarrow{L^\infty(\mathbb{P})} 0 \Rightarrow G_t(\xi^n) \xrightarrow{\mathbb{P}} G_t(\xi) \textit{ for each } t \in [0, T]$$

holds for simple ξ^n and ξ if and only if the adapted càdlàg process X is a so-called semimartingale, i.e., of the form

$$X_t = M_t + A_t, \quad t \in [0, T],$$

for a local martingale M and an adapted process A with right-continuous paths of bounded variation.

Upshot: A model X for discounted asset prices has to be specified as a semimartingale in order to have stability of gains processes in the above (mild?) sense.

Processes of bounded variation

A càdlàg process A is of bounded variation if almost surely

$$TV_T(A) := \sup_{0=t_0\leq t_1\leq\cdots\leq t_n=T,\ n\in\mathbb{N}} \sum_{i=1}^{n} |A_{t_i} - A_{t_{i-1}}| < \infty.$$

Lemma

A càdlàg process A is of bounded variation if and only if it can be written as the difference

$$A_t = A_t^{\uparrow} - A_t^{\downarrow}, \quad t \in [0, T],$$

of two nondecreasing càdlàg processes $A^{\uparrow}$, $A^{\downarrow}$. These are uniquely determined if we insist on the minimality requirement $TV_T(A) = A_T^{\uparrow} + A_T^{\downarrow}$ (Hahn-decomposition).

More on this in our exercise class.

(Local) martingales

Recall: A process (M_t) is a martingale, if $M_t \in L^1(\mathbb{P})$ for each t with

$$\mathbb{E}\left[M_t \middle| \mathscr{F}_s \right] = M_s \text{ for any } s \in [0, t].$$

Definition

A process (M_t) is a *local martingale* if it is locally a martingale in the sense that there is an increasing sequence of stopping times $T^n \nearrow \infty$, a so-called *localizing sequence*, such that each stopped process

$$M^{T^n} = (M_t^{T^n}) \text{ with } M_t^{T^n}(\omega) := M_{T^n(\omega) \wedge t}(\omega), \quad n = 1, 2, \ldots,$$

is a martingale. The class of local martingales will be denoted by $\mathscr{M}_{\mathrm{loc}}$; continuous local martingales are collected in $\mathscr{M}_{\mathrm{c,loc}}$.

Remark: We will later get to see local martingales which are not martingales (i.e., *strict* local martingales).

Stochastic processes in continuous time

Technical problem:

? *Is M_T for a finite stopping time T actually measurable with respect to $\mathscr{F}_T$?*

! Yes, but we need M to have right-continuous (càd) paths and also $(\mathscr{F}_t)$ to be right-continuous.

Indeed: For right-continuous M we have

$$M_T = \lim_n \sum_k \underbrace{M_{\frac{k+1}{n}} 1_{[\frac{k}{n},\frac{k+1}{n})}(T)}_{\in \mathscr{F}_{T+\frac{1}{n}}} \in \bigcap_n \mathscr{F}_{T+\frac{1}{n}} =: \mathscr{F}_{T+} \overset{!}{=} \mathscr{F}_T .$$

Fortunately, under the "usual hypotheses" of right-continuity and completeness of the filtration we can assume martingales to be right-continuous without too much loss ...

Modifications and indistinguishability

Lemma

Under the usual hypothesis, any local martingale M has a càdlàg modification, i.e., there is $\tilde{M}$ with càdlàg paths and

$$\mathbb{P}[M_t = \tilde{M}_t] = 1 \text{ at any time } t.$$

This modification is unique up to indistinguishability in the sense that any alternative $\tilde{M}'$ satisfies

$$\mathbb{P}[\tilde{M}_t = \tilde{M}'_t \text{ at any time } t] = 1.$$

! Note that it is not even clear whether the set $\{\omega \in \Omega : M_t(\omega) = \tilde{M}_t(\omega) \text{ for all } t\}$ is measurable for a not necessarily càdlàg martingale M. So talking about indistinguishability of M from another process $\tilde{M}$ will typically only make sense after choosing the above nice version.

Proof of uniqueness up to indistinguishability

Word of caution

! Beware of statements almost surely involving uncountably many statements because exceptional nullsets may pile up to a no longer negligible set.

Good news: We leave these intricacies for courses on stochastic analysis to sort out and in fact will make sure that we will most of the time only deal with processes which even have continuous sample paths (almost surely).

Outline

PnL and integration

? *How to exend the definition of*

$$G_t(\xi) = \sum_{i=1}^{n} \xi_{T_i}(X_{T_i \wedge t} - X_{T_{i-1} \wedge t}) =: \int_0^t \xi_s dX_s$$

to continually changing ξ*?*

! We need to define integrals

$$\int_0^t \xi_s dX_s$$

beyond simple integrands.

But: This is easier said than done in a relevant way for financial mathematics as will transpire from our next result!

Pathwise integration almost incompatible with no arbitrage

Lemma

If we insist on defining the stochastic integral pathwise in the sense that for ξ^n, ξ *with* $|\xi^n| \leq 1, |\xi| \leq 1$ *we have*

$$G_t(\xi^n) \xrightarrow{\mathbb{P}} G_t(\xi) \text{ for each } t \in [0, T] \text{ on } \left\{ \sup_{t \in [0,T]} |\xi^n_t - \xi_t| \to 0 \right\}$$

then X *must be of bounded variation:* $X = 0 + A$*, and in this case the Lebesgue-Stieltjes integral*

$$G_t(\xi) := \int_0^t \xi_s dA_s := \int_{[0,t]} \xi_s \mu(ds), \quad t \in [0, T],$$

w.r.t. to the signed measure μ *with "distribution function"* $\mu([0, s]) = A_s$, $s \in [0, T]$*, will yield the extension.*

Upshot: Defining a PnL pathwise when stock prices fluctuate like for a (geometric) Brownian motion is impossible.

Proof

Via Banach-Steinhaus Theorem: Exercise on first problem set.

If we dispense with this stability, we will still want no arbitrage—and again are led to semimartingales

The stability requirement of Bichteler-Dellacherie is mathematically convenient, but not the only one conceivable from a financial-economic point of view.

This is very different for the notion of arbitrage though. Remarkably Delbaen and Schachermayer (1994) show that

mildly more than "no arbitrage" implies X is a semimartingale

More precisely, the condition "no free lunch with vanishing risk for simple integrands" suffices to conclude that X is a semimartingale:

$$L^\infty\text{-closure}\{g \in L^\infty : g \leq G_T(\xi) \text{ for a simple } \xi\} \cap L^\infty_+ = \{0\}.$$

Upshot: Wouldn't it be lovely if we could integrate with respect to local martingales M as well?

Outline

The space of square integrable martingales

Lemma

The set

$$\mathscr{M}^2 := \{M \ : \ M \textit{ càdlàg martingale with } \mathbb{E}[M_t^2] < \infty, \ t \in [0,T]\}$$

is a Hilbert-space with scalar product and norm given by

$$(M,N)_{\mathscr{M}^2} := \mathbb{E}[M_T N_T] \textit{ and } \|M\|_{\mathscr{M}^2} := \|M_T\|_{L^2(\mathbb{P})}.$$

We have the equivalence of norms

$$\|M\|_{\mathscr{M}^2} \leq \|\sup_{t\in[0,T]} |M_t|\|_{L^2(\mathbb{P})} \leq 2\|M\|_{\mathscr{M}^2} \textit{ for any } M \in \mathscr{M}^2$$

and the class

$$\mathscr{M}_c^2 := \{M \ : \ M \textit{ is a square-integrable continuous martingale}\}$$

is a closed subspace of $\mathscr{M}^2$*.*

Proof

Quadratic variation

? *For $M \in \mathscr{M}^2$, what is the compensator of the submartingale M^2?*

Theorem

For any $M \in \mathscr{M}^2$, there is an adapted, right-continuous, increasing process $[M]$ such that for any refining sequence of stopping time partitions

$$\tau^N = \{0 = T_0^N \leq \cdots \leq T_{n_N}^N = T\}, \quad N = 1, 2, \ldots,$$

with smaller and smaller mesh size

$$\|\tau^N\| := \sup_{i=1,\ldots,n_N} |T_i^N - T_{i-1}^N| \xrightarrow{\mathbb{P}} 0.$$

we have

$$\sup_{t\in[0,T]} \left| \sum_{i=1}^{n_N} (M_{T_i^N \wedge t} - M_{T_{i-1}^N \wedge t})^2 - [M]_t \right| \xrightarrow{\mathbb{P}} 0.$$

Quadratic variation (ctd.)

Theorem (ctd.)

Moreover, the process $[M]$ *compensates* M^2 *in the sense that* $M^2 - [M]$ *is a martingale and we have*

$$\|M\|_{\mathscr{M}^2}^2 = \|M_0\|_{L^2(\mathbb{P})}^2 + \mathbb{E}\left[\, [M]_T \right].$$

Finally, if M *is continuous, so is* $[M]$*.*

Definition

The process $[M]$ of the above theorem is called the *quadratic variation* of M.

Remark

Our discussion of Brownian motion $M = W$ last term has established that its quadratic variation is $[M]_t = t$.

Proof postponed

The proof is postponed as it draws on ideas which we will use in greater generality (and arguably greater transparency) in our construction of stochastic integrals.

Even then we will only deal with the case of a continuous local martingale. As we will see at the end of this section, it essentially amounts to the construction of our first nontrivial stochastic integral, namely $\int_0^t M_{s-} dM_s$.

Back to our extension problem

We need to extend the definition of $I(\xi) := \int_0^\cdot \xi_s dM_s$ beyond simple integrands

$$\xi_s = \sum_{i=1}^n \xi_{T_i} 1_{(T_{i-1},T_i]}(s), \quad s \in [0, T],$$

with stopping times $0 = T_0 \leq T_1 \leq \cdots \leq T_n = T$ and $\xi_{T_i} \in \mathscr{F}_{T_{i-1}}$, for which we keep insisting on

$$I_t(\xi) := I_t^M(\xi) := \sum_{i=1}^n \xi_{T_i}(M_{T_i \wedge t} - M_{T_{i-1} \wedge t}), \quad t \in [0, T].$$

This will become possible by two key observations on the structure of such integrals which will hold for integrands in

$$\mathscr{S} := \{\xi \text{ is of the above simple form and bounded}\}.$$

Key observation I: Martingale property preserved

Lemma (Preservation of martingale property)

If M is a martingale and $\xi \in \mathscr{S}$ is simple and bounded, also $I^M(\xi) = \int_0^\cdot \xi_s dM_s$ is a martingale.

Remark

This can be viewed as a continuous-time version of Doob's system theorem and, in fact, its proof essentially amounts to just that.

Proof

Key observation II: Ito's isometry

Lemma (Ito isometry)

For square-integrable $M \in \mathscr{M}^2$ *and bounded, simple* ξ*, also* $I^M(\xi) = \int_0^\cdot \xi_s dM_s$ *is in* $\mathscr{M}^2$*, and we have the isometry*

$$\|I^M(\xi)\|_{\mathscr{M}^2} = \| \int_0^T \xi_s dM_s \|_{L^2(\mathbb{P})} \stackrel{!}{=} \|\xi\|_{L^2(\mathbb{P}\otimes d[M])} := \mathbb{E}[\int_0^T \xi_s^2 d[M]_s]^{1/2}.$$

Remark

The measure $\mathbb{P} \otimes d[M]$ is the measure on $\Omega \times [0, T]$ defined by

$$(\mathbb{P} \otimes d[M])[A] := \int_\Omega \int_0^T 1_A(\omega, s) d[M]_s(\omega) \mathbb{P}(d\omega), \ A \in \mathscr{F}_T \otimes \mathscr{B}([0, T]).$$

This measure has finite total mass

$$(\mathbb{P} \otimes d[M])(\Omega \times [0, T]) = \mathbb{E}[\ [M]_T] = \mathbb{E}[(M_T - M_0)^2] < \infty.$$

Proof

Isometric extension of the stochastic integral

Theorem

For $M \in \mathscr{M}^2$, the mapping

$$\mathscr{S} \ni \xi \mapsto I^M(\xi) = \int_0^{\cdot} \xi_s dM_s \in \mathscr{M}^2$$

has a unique continuous (and linear) extension to

$$L^2(M) := L^2(\mathbb{P} \otimes d[M])\text{-closure}(\mathscr{S}) = L^2(\Omega \times [0,T], \mathscr{P}, \mathbb{P} \otimes d[M])$$

where $\mathscr{P} := \sigma(\mathscr{S})$ is the so-called predictable σ-field.
For $\xi \in L^2(M)$, $I^M(\xi) = \int_0^{\cdot} \xi_s dM_s \in \mathscr{M}^2$ is a square-integrable martingale satisfying Ito's isometry

$$\|I^M(\xi)\|_{\mathscr{M}^2} = \|\int_0^T \xi_s dM_s\|_{L^2(\mathbb{P})} = \|\xi\|_{L^2(\mathbb{P}\otimes d[M])} = \mathbb{E}[\int_0^T \xi_s^2 d[M]_s]^{1/2}.$$

Finally, if $M \in \mathscr{M}_c^2$ is continuous, so is $I^M(\xi)$ for $\xi \in L^2(M)$.

Proof

Definition of the stochastic integral

Definition

For $\xi \in L^2(M)$ and $M \in \mathscr{M}^2$, the process $I^M(\xi) = \int_0^{\cdot} \xi_s dM_s \in \mathscr{M}^2$ is called the *stochastic integral* of ξ w.r.t. M.

Remark

Note that we have defined this integral *not in a pathwise manner*, but by a limit in the Hilbert-Space $\mathscr{M}^2 \approx L^2(\mathbb{P})$ and so it seemingly depends on the measure $\mathbb{P}$. Indeed, it will later be important to understand that this is not really the case, at least up to equivalent changes of measure.

Right now of course, any attempt at this hits a road block right away when we observe that the martingale property of the integrator $M \in \mathscr{M}^2 = \mathscr{M}^2(\mathbb{P})$ obviously depends strongly on $\mathbb{P}$. So, we need to be able to integrate against more than martingales.

Construction of quadratic variation for a continuous L^2-martingale

Theorem

For $M \in \mathscr{M}_c^2$*, the sequence of stopping time partitions* $\bar{\tau}^N = \{0 = \bar{T}_0^N \le \bar{T}_1^N \le \cdots \le T\}$ *given for* $N = 1, 2, \ldots$ *by*

$$\bar{T}_0^N := 0, \quad \bar{T}_i^N := \inf\{t \ge \bar{T}_{i-1}^N : |M_t - M_{\bar{T}_{i-1}^N}| \ge 2^{-N}\} \wedge T, \; i = 1, 2, \ldots$$

is such that there is an adapted, continuous, increasing process $[M]$ *with*

$$\sup_{t \in [0,T]} \left| \sum_{i=1,2,\ldots} (M_{\bar{T}_i^N \wedge t} - M_{\bar{T}_{i-1}^N \wedge t})^2 - [M]_t \right| \xrightarrow{\mathbb{P}} 0.$$

Moreover, $M^2 - [M]$ *is a continuous martingale and we have*

$$\|M\|_{\mathscr{M}^2}^2 = \|M_0\|_{L^2(\mathbb{P})}^2 + \mathbb{E}[\,[M]_T].$$

Proof

Observation

An inspection of the above proof, shows that the construction of $[M]$ is accomplished via a "hands on" construction of the stochastic integral $I_t =$"$\int_0^t M_{s-} dM_s$"$= \frac{1}{2}(M_t^2 - [M]_t)$ (i.e., independently from our general construction above, but of course using some of the key ideas we also use there).

The need to construct an iterated integral like $\int_0^\cdot M_{s-} dM_s$ to develop an integration theory is *not at all coincidental*, but in fact the key insight behind what is called rough path integration theory as discovered and developed by Terry Lyons since the 90s. This field has since seen tremendous growth and actually spurred the theory of regularity structures for which Martin Hairer was awarded the Fields Medal in 2014.

Outline

Convenient localization of continuous local martingales

By definition, a local martingale $M \in \mathscr{M}_{\text{loc}}$ has a localizing sequence of stopping times (T_n) such that M^{T_n} is a martingale. But it may not have a localizing sequence that ensures even $M^{T_n} \in \mathscr{M}^2$—and so extending our stochastic integral by localization becomes rather tricky in general.

Fortunately, things are a lot easier for *continuous* local martingales $M \in \mathscr{M}_{\text{c,loc}}$ because for these we can choose

$$T_n := \inf\{t \ : \ |M_t| \geq n\}, \quad n = 1, 2, \ldots,$$

for which M^{T_n} is even uniformly bounded: $|M_t^{T_n}| \leq n$, $t \in [0, T]$.

We therefore will confine ourselves to extending the stochastic integral to integrators $M \in \mathscr{M}_{\text{c,loc}}$.

Quadratic variation of continuous local martingales

Theorem

For a continuous local martingale $M \in \mathscr{M}_{c,loc}$, we can consistently define $[M]$ on $[0, T]$ by putting

$$[M]_t := [M^{T_n}]_t \text{ on } \{t \leq T_n\}, \quad n = 1, 2, \ldots,$$

where $(T_n)_{n=1,2,\ldots}$ is any localizing sequence with $M^{T_n} \in \mathscr{M}_c^2$. Moreover, $[M]$ is the unique continuous, adapted increasing process starting in $[M]_0 = 0$ for which $M^2 - [M]$ is a continuous local martingale.

Definition

The process $[M]$ is called the *quadratic variation* of the continuous local martingale $M \in \mathscr{M}_{\text{c,loc}}$.

Example

For a Brownian motion W we have $[W]_t = t$.

Proof

Lemma

If a continuous local martingale $M \in \mathscr{M}_{c,loc}$ *has finite variation* $TV_T(M) < \infty$*, then it is constant:*

$$M_t \equiv M_0, t \in [0, T].$$

Proof

Relation between $[M]$ and $\langle M \rangle$

Remark

As a continuous, adapted process, $[M]$ for $M \in \mathscr{M}_{\mathrm{c,loc}}$ is predictable. It thus coincides with what is known as the *predictable quadratic variation* $\langle M \rangle$ which is defined as the unique *predictable*, increasing process A starting in 0 such that $M^2 - A \in \mathscr{M}_{\mathrm{loc}}$—if such a process exists. Existence of such an $A = \langle M \rangle$ can be shown for some merely càdlàg martingales, for which, however, it typically differs from $[M]$.

Fortunately, we will not have to worry about any of that because we will (essentially) always deal with continuous processes and so both bracket process

$$[M] = \langle M \rangle$$

can be used interchangeably.

Proof

Stochastic integration w.r.t. continuous local martingales

Theorem

For $M \in \mathscr{M}_{c,loc}$ *and* $\xi \in L^2(M) := L^2(\mathbb{P} \otimes d[M])$*, letting*

$$I_t^M(\xi) := \int_0^t \xi_s dM_s := \int_0^t \xi_s dM_s^{T_n} \text{ on } \{t \leq T_n\}, \quad n = 1, 2, \ldots,$$

yields a linear, isometric mapping

$$L^2(M) \ni \xi \mapsto \int_0^\cdot \xi_s dM_s. \in \mathscr{M}_c^2$$

For integrands in $L(M) := \{\xi$ *predictable with* $\int_0^T \xi_s^2 d[M]_s < \infty\}$*, the above recipe still yields* $I^M(\xi) = \int_0^\cdot \xi_s dM_s \in \mathscr{M}_{c,loc}$.

Definition

The process $I^M(\xi) = \int_0^\cdot \xi_s dM_s$ is called the stochastic integral of $\xi \in L(M)$ w.r.t. $M \in \mathscr{M}_{c,loc}$.

Proof

Quadratic variation of a stochastic integral

Theorem

For $M \in \mathscr{M}_{c,loc}$ and $\xi \in L(M)$, the quadratic variation of $\int_0^\cdot \xi_s dM_s \in \mathscr{M}_{c,loc}$ is given by

$$[\int_0^\cdot \xi_s dM_s]_t = \int_0^t \xi_s^2 d[M]_s, \quad t \in [0, T].$$

Proof

Continuous semimartingales

Let us now patch together our integrals w.r.t. bounded variation processes A and w.r.t. local martingales M. This will be particularly fruitful for the following class:

Definition

A process S is called a *continuous semimartingale* if it can be written in the form $S = M + A$ for a continuous local martingale M and a continuous adapted process A of bounded variation.

Remark

For a semimartingale with continuous paths, it can be shown that it is also a continuous semimartingale in the above sense.

Lemma

The continuous processes M and A in the decomposition of a continuous semimartingale are unique after normalizing to $A_0 = 0$.

Definition

The decomposition is called the *Doob-Meyer decomposition* of S.

Proof

Brownian motion—a lame example

Definition

An adapted stochastic process W on $(\Omega, (\mathscr{F}_t), \mathbb{P})$ is called a *Brownian motion* or *Wiener process* if it has continuous sample paths $t \mapsto W_t(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, starts at zero $\mathbb{P}[W_0 = 0] = 1$ and has independent Gaussian increments in the sense that $\mathbb{P}[W_t - W_s \in dx | \mathscr{F}_s] = N(0, t-s)(dx)$ for any $0 \leq s \leq t$.

Of course, Brownian motion is in particular a continuous martingale (actually in a sense discussed later sort of the only one) and, thus, also a continuous semimartingale with decomposition $S = W + 0$.

In fact, any Ito-process is a continuous semimartingale as well—this should be clear from FiMa I.

Brownian motion—a lame example?

Let us consider a Brownian motion W on $(\Omega, (\mathscr{F}_t), \mathbb{P})$ and assume an “insider” has right from the start access to additional information, namely to its value at time $T > 0$. So the relevant filtration for this insider is

$$\mathscr{G}_t := \mathscr{F}_t \vee \sigma(W_T), \quad t \geq 0.$$

Exercise

Clearly, W is *not* a Brownian motion on $(\Omega, (\mathscr{G}_t), \mathbb{P})$, but it remains a continuous semimartingale with decomposition

$$W_t = W'_t + \int_0^{t \wedge T} \frac{W_T - W_s}{T - s} ds$$

where W' *is* a $(\Omega, (\mathscr{G}_t), \mathbb{P})$-Brownian motion.

Remark

On $(\Omega, (\mathscr{G}_t), \mathbb{P})$, W is called a *Brownian bridge* over $[0, T]$.

Brownian motion—a lame example??

Let us consider a Brownian motion W on $(\Omega, (\mathscr{F}_t), \mathbb{P})$ and assume an "insider" has right from the start access to additional information, namely to its path up to time $T > 0$. So the relevant filtration for this insider is

$$\mathscr{G}_t := \mathscr{F}_t \vee \sigma(W_s,\ s \leq T), \quad t \geq 0.$$

Exercise

Clearly, W is *not* a Brownian motion on $(\Omega, (\mathscr{G}_t), \mathbb{P})$, and it is not even a continuous semimartingale.

Remark

Hence, the notion of a semimartingale is quite fickle when it comes to tampering with the filtration.

Fortunately, when it comes to tampering with the probability measure, the notion of a semimartingale is rather stable at least if we confine ourselves to equivalent changes of measure; see our discussion below on Grisanov's theorem.

Stochastic integration w.r.t. continuous semimartingales

For a continuous semimartingale with Doob-Meyer decomposition $S = M + A$, we can now define $\int_0^\cdot \xi_s dS_s$ for integrands from

$$L(S) := \left\{ \xi \text{ predictable with } \int_0^T |\xi_s|^2 d[M]_s + \int_0^T |\xi_s||dA_s| < \infty \right\}$$

as the continuous semimartingale with Doob-Meyer decomposition

$$\int_0^t \xi_s dS_s := \int_0^t \xi_s dM_s + \int_0^t \xi_s dA_s, \quad t \in [0, T].$$

Remark

If A is continuous with bounded total variation $TV_T(A) < \infty$, its Hahn-decomposition $A = A^\uparrow - A^\downarrow$ yields minimal continuous, increasing $A^\uparrow$, $A^\downarrow$ with $A^\uparrow_T + A^\downarrow_T = TV_T(A)$. We use these to define

$$\int_0^t |\xi_s||dA_s| := \int_0^t |\xi_s|(dA^\uparrow_s + dA^\downarrow_s), \quad t \in [0, T].$$

Limit theorem for stochastic integrals

Continuous dependence of integrals on integrands can also be ensured for integration with respect to semimartingales:

Theorem

For a continuous semimartingale $S = M + A$ *and* ξ, $\xi^n \in L(S)$, $n = 1, 2, \dots$, *with*

$$\int_0^T |\xi_s - \xi_s^n|^2 d[M]_s + \int_0^T |\xi_s - \xi_s^n||dA_s| \xrightarrow{\mathbb{P}} 0,$$

we have $\int_0^\cdot \xi_s^n dS_s \to \int_0^\cdot \xi_s dS_s$ *uniformly in probability:*

$$\sup_{t\in[0,T]} \left| \int_0^t \xi_s^n dS_s - \int_0^t \xi_s dS_s \right| \xrightarrow{\mathbb{P}} 0.$$

Upshot: Suitable criteria for classical Lebesgue-style limit theorems can be used for limits of stochastic integrals.

Proof

Finally: Proof of our theorem on quadratic variation

Corollary

Consider a refining sequence of stopping time partitions

$$\tau^N = \{0 = T_0^N \leq \cdots \leq T_{n_N}^N = T\} \text{ with } \|\tau^N\| := \sup_{i=1,\dots,n_N} |T_i^N - T_{i-1}^N| \xrightarrow{\mathbb{P}} 0.$$

Then, for any $M \in \mathscr{M}_{c,loc}$*, we have*

$$\sup_{t\in[0,T]} \left| \sum_{i=1}^{n_N} (M_{T_i^N \wedge t} - M_{T_{i-1}^N \wedge t})^2 - [M]_t \right| \xrightarrow{\mathbb{P}} 0.$$

In particular, the special partition sequence $\bar{\tau}^N$ *considered in our construction of quadratic variation* $[M]$ *for* $M \in \mathscr{M}_c$ *does not matter after all.*

Proof

Integration w.r.t. stochastic integrals

Sometimes we will want to integrate processes ξ' w.r.t. a semimartingale S' which is itself a stochastic integral, i.e., $S' = \int_0^{\cdot} \xi_s dS_s$ for some continuous semimartingale S and $\xi \in L(S)$:

Lemma

In the above situation, we have $\xi' \in L(S')$ iff $\xi'\xi \in L(S)$ and in this case

$$\int_0^t \xi'_s dS'_s = \int_0^t \xi'_s \xi_s dS_s, \quad t \in [0, T].$$

Remark

This lemma gives formal meaning to and justification of statements like $dS'_s = \xi_s dS_s$ which we will make from time to time.

Proof

Finally, finally back to Finance

Corollary

For any continuous semimartingale X *describing an asset's discounted price flucutations, we can define the discounted profits and losses from trading according to* $\xi \in L(X)$ *by putting*

$$G_t(\xi) := \int_0^t \xi_s dX_s, \quad t \in [0, T].$$

Supplementing ξ *in a self-financing manner with* ξ^0 *to a strategy* $\bar{\xi} = (\xi^0, \xi)$ *with initial value* $v = \bar{\xi}_0 \bar{X}_0$ *leads to the wealth process*

$$V_t^{\bar{\xi}} := \xi_t^0 + \xi_t X_t := V_t^{v,\xi} := v + \int_0^t \xi_s dX_s, \quad t \in [0, T].$$

This yields the desired consistent and stable definition of the wealth dynamics beyond simple startegies.

MISSION MORE THAN ACCOMPLISHED!

Outline

Financial modeling means constructing semimartingales

A consequence from the last chapter is that we will want to model discounted asset prices as (continuous) semimartingales:

$$X = M + A$$

where

- A is a continuous adapted process of bounded variation specifying the local price trend:

  $$dA_t = \text{“average price change over the next } dt\text{-period”}$$

- M is a continuous local martingale specifying (sort of) symmetric price fluctuations around this trend:

  $$dM_t = \text{“conditionally centered random noise over next } dt\text{-period”}$$

Obvious example: Ito-processes

$$X_t = X_0 + \int_0^t \Upsilon_s dW_s + \int_0^t \eta_s ds, \quad t \in [0, T].$$

New from old: smooth transformations of semimartingales

For Ito-processes we found that they are stable with respect to smooth transformations because of Ito's formula.

? *Does Ito's formula also work for continuous semimartingales?*

! Yes, and we can use it to work out the continuous local martingale part and the continuous bounded variation part for any process emerging as a smooth transformation of a continuous semimartingale.

Remark

So rather than having to compute market trends of transformed quantities by taking expectations, we will be able to determine them by calculus—which is a lot simpler!
Similarly, we will be able to understand how symmetric fluctuations are transformed, again by simple calculus.

Outline

Ito's formula

Theorem

For a continuous semimartingale $S = M + A$ *and* $f \in C^2(\mathbb{R})$*, also* $f(S)$ *is a continuous semimartingale and we have*

$$f(S_t) - f(S_0) = \int_0^t f'(S_s) dS_s + \frac{1}{2}\int_0^t f''(S_s) d[S]_s$$
$$= \underbrace{\int_0^t f'(S_s) dM_s}_{\text{local martingale part}}$$
$$+ \underbrace{\int_0^t f'(S_s) dA_s + \frac{1}{2}\int_0^t f''(S_s) d[M]_s}_{\text{bounded variation part}}.$$

Remark

This is often written in differential form:

$$df(S_t) = f'(S_t) dS_t + \frac{1}{2} f''(S_t) d[S]_t$$

Illustration

Remember how we constructed the quadratic variation for a continuous martingale M with $M_0 = 0$ essentially by constructing an iterated integral and finding that

$$\int_0^t M_s dM_s = \frac{1}{2}(M_t^2 - [M]_t), \quad t \in [0, T].$$

We can now view this also as a first instance of Ito's formula for the special case where $f(x) := \frac{1}{2}x^2$:

$$\frac{1}{2}M_t^2 = \int_0^t M_s dM_s + \frac{1}{2}\int_0^t 1 d[M]_s, \quad t \in [0, T].$$

Quadratic variation of a continuous semimartingale

Lemma

For a continuous semimartingale $S = M + A$*, we have*

$$\sup_{t\in[0,T]} \left| \sum_{i=1}^{n_N} (S_{T_i^N \wedge t} - S_{T_{i-1}^N \wedge t})^2 - [M]_t \right| \xrightarrow{\mathbb{P}} 0.$$

for any sequence of stopping time partitions

$$\tau^N = \{0 = T_0^N \leq \cdots \leq T_{n_N}^N = T\} \text{ with } \|\tau^N\| := \sup_{i=1,\ldots,n_N} |T_i^N - T_{i-1}^N| \xrightarrow{\mathbb{P}} 0.$$

Definition

The process $[S] := [M]$ is called the quadratic variation of the continuous semimartingale $S = M + A$.

Remark

Obviously, $S^2 - [S] = S^2 - [M]$ will typically not be a continuous local martingale, unless already S is one.

Proof

Proof of Ito's formula via Tayloring telescoping sums

Going to higher dimensions

In order to extend the preceding proof to functions $f = f(S^1, S^2)$ of two (or more) continuous semimartingales, we need to do a multivariate Taylor approximation and then understand the limits of sums as in the following lemma.

Lemma

Consider a sequence of stopping time partitions

$$\tau^N = \{0 = T_0^N \leq \cdots \leq T_{n_N}^N = T\} \text{ with } \|\tau^N\| := \sup_{i=1,\dots,n_N} |T_i^N - T_{i-1}^N| \stackrel{\mathbb{P}}{\to} 0.$$

For two continuous semimartingales S^1, S^2, the sums

$$\sum_{i=1}^{n_N} (S^1_{T_i^N \wedge t} - S^1_{T_{i-1}^N \wedge t})(S^2_{T_i^N \wedge t} - S^2_{T_{i-1}^N \wedge t})$$

converge uniformly in $t \in [0, T]$ in probability to a continuous adapted process of bounded variation denoted $[S^1, S^2]$.

Covariation of continuous semimartingales

Definition

The process $[S^1, S^2]$ of the preceding lemma is called the *quadratic covariation* of S^1 and S^2.

Lemma

With obvious notation we have $[S^1, S^2] = [M^1, M^2]$*, and* $[M^1, M^2]$ *is the only continuous adapted process of bounded variation* A *for which* $M^1 M^2 - A$ *is a continuous local martingale.*

Corollary

For two independent Brownian motions W^1, W^2*, we have*

$$[W^1, W^2]_t = 0, \quad t \in [0, T].$$

Proof: $W^1 W^2$ is already a continuous martingale (?!) for independent Brownian motions W^1, W^2.

Covariation of stochastic integrals

Just like the quadratic variation of a stochastic integral can be computed from its integrand and integrator, also the covariation between such integrals can be obtained from its inputs:

Lemma

For $\xi \in L(S)$ and $\xi' \in L(S')$, we have $\xi\xi' \in L([S, S'])$ because of the Kunita-Watanabe inequality

$$\int_0^T |\xi_s||\xi'_s||d[S, S']_s| \leq \left(\int_0^T \xi_s^2 d[S]_s\right)^{1/2} \left(\int_0^T \xi_s'^2 d[S']_s\right)^{1/2}.$$

Moreover, we can compute the covariation

$$[\int_0^{\cdot} \xi dS, \int_0^{\cdot} \xi' dS']_t = \int_0^t \xi_s \xi'_s d[S, S']_s, \quad t \in [0, T].$$

Proof

Proof of the preceding lemma

By polarization, the statements on covariations can be reduced to properties of quadratic variation.

Multivariate Ito formula

Theorem

Let $S = (S^1, \dots, S^d)$ be a vector of d continuous semimartingales (i.e., a d-dimensional semimartingale). Then, for any $f \in C^2(\mathbb{R}^d)$ also $f(S)$ is a continuous semimartingale and we have

$$
\begin{aligned}
f(S_t) - f(S_0) &= \int_0^t \nabla f(S_s) \cdot dS_s + \frac{1}{2} \int_0^t \nabla^2 f(S_s) \cdot d[S]_s \\
&= \sum_{i=1}^d \int_0^t \partial_i f(S_s) dS_s^i \\
&\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 f(S_s) d[S^i, S^j]_s
\end{aligned}
$$

for $t \in [0, T]$.

Continuous semimartingales form an algebra

Theorem

The sums and products of continuous semimartingales yield again continuous semimartingales.
Specifically, we have Ito's product rule

$$S_t^1 S_t^2 = S_0^1 S_0^2 + \int_0^t S_s^1 dS_s^2 + \int_0^t S_s^2 dS_s^1 + [S^1, S^2]_t, \quad t \in [0, T],$$

for any pair of continuous semimartingales S^1, S^2.

Remark

Ito's product rule could also have been established right from the construction of quadratic covariation, i.e., without using Ito's formula, just using stochastic integration ...

Proof

Another proof of Ito's formula via polynomial approximation

Exactly as for Ito processes.

Outline

Linear Stochastic Differential Equations

Recall how for the Black-Scholes model

$$X_t = s_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right), \quad t \in [0, T],$$

it was helpful to write its dynamics as solution to the linear SDE

$$X_0 = s_0, \quad dX_t = X_t(\sigma dW_t + (\mu - r)dt).$$

? *Which continuous semimartingales can we view as such a solution to a linear SDE?*

Stochastic logarithm and exponential

Theorem

For any continuous semimartingale L, there exists a unique (up to indistinguishability) solution X to

$$X_0 = x_0, \quad dX_t = X_t dL_t,$$

namely $X_t = x_0 \mathscr{E}(L)_t$, where

$$\mathscr{E}(L)_t =$$

denotes the so-called stochastic exponential of L.
Conversely, any strictly positive continuous semimartingale X can be written as $X = x_0 \mathscr{E}(L)$ if we choose L as the so-called stochastic logarithm of X:

$$L_t = \int_0^t \frac{dX_s}{X_s}.$$

Proof

Black-Scholes as an exponential

So, we can write the Black-Scholes model

$$X_t = s_0 \exp(\sigma W_t + (\mu - r - \frac{1}{2}\sigma^2)t), \quad t \in [0, T]$$

as the stochastic exponential of Brownian motion with volatility σ and drift $\mu - r$:

$$X_t = s_0 \mathscr{E}(L)_t \text{ for } L_t = \sigma W_t + (\mu - r)t, \quad t \in [0, T].$$

Outline

Outline

Jouer à la martingale ...

In a model like, say, Black-Scholes consider the following strategy defined inductively over $[0, T]$:

$n = 1$: Buy and hold one unique of stock $\xi_1 = 1$ until you have gained $1\$$, but at most until time $\frac{1}{2}T$ is left.

$n \rightsquigarrow n+1$: Once you have reached the $1\$$ gain, you walk away with your win; if by time $T_n := (1 - 2^{-n})T$, this hasn't happened, you choose your next position $\xi_{n+1} > 0$ in such a way that

$$\mathbb{P}\Bigg[\underbrace{\sup_{t \in [T_n, T_{n+1}]} \{V_{T_n} + \xi_{n+1}(X_t - X_{T_n})\} \geq 1}_{\text{"walk away with 1\$ between } T_n \text{ and } T_{n+1}\text{"}} \Bigg| \mathscr{F}_{T_n}\Bigg] \geq 1/2$$

and try your luck over $[T_n, T_{n+1}]$.

Jouer à la martingale . . . (ctd.)

Choosing $\xi_{n+1} \in \mathscr{F}_{T_n}$ as required is possible because the above probability coincides with

$$\mathbb{P}\left[\frac{1 - V_{T_n}}{\sup_{t\in[T_n, T_{n+1}]}\{X_t - X_{T_n}\}} \leq x \middle| \mathscr{F}_{T_n} \right] \text{ for } x = \xi_{n+1} \in \mathscr{F}_{T_n}$$

which will converge to 1 as $x \uparrow \infty$ (unless X stops fluctuating over $[T_n, T_{n+1}]$ which obviously is not the case for the Black-Scholes model and actually for any other reasonable model (?!?)).

Lemma

The above recipe yields a selffinancing strategy which, starting from zero initial capital, ends up with terminal discounted wealth

$$V_T = 1 \quad \mathbb{P}\text{-a.s.},$$

and actually accomplishes this feat by an almost surely finite number of transactions.

Proof

Arbitrage!?

So the strategy described above will produce a riskless profit in the Black-Scholes model and, in fact, in any typical continuous-time model!

- ? *What to do? Should we better stop looking at continuous-time models??*
- ! No! We should just rule out strategies which proceed as described above—and we have good reason to do so, because, while ending up at 1$ a.s. in the end, the PnL inbetween cannot be bounded from below a priori. So a strategy like above is as infeasible as is a **doubling strategy** at a roulette table: we just won't have the financial backing to guarantee our solvency while following it.

Remark

The reckless "winning" stragegy is known in French as "jouer à la martingale", one of the origins for the notion "martingale".

Doubling strategies tried — and failed ...

There are quite a few examples of "doubling strategies" gone bad:

- Nick Leeson bringing down Barings Bank in 1995; https://en.wikipedia.org/wiki/Nick_Leeson
- Jérôme Kerviel causing a loss of 4.9bn for Société Générale in 2008; https://en.wikipedia.org/wiki/Jerome_Kerviel
- Bruno Iksil losing 6.2bn for JP Morgan in 2012; https://en.wikipedia.org/wiki/2012_JPMorgan_Chase_trading_loss

A common feature seems to be that early "successful" runs of the strategy where followed by the one run too many — all of this enabled by circumventing ineffective risk-management systems.

Admissible strategies

Definition

A selffinancing strategy $\bar{\xi} = (\xi^0, \xi)$ is called *admissible* if its discounted PnL is bounded from below over $[0, T]$, i.e., if there is a constant $c \in (-\infty, 0]$ such that

$$G_t(\xi) = \int_0^t \xi_s dX_s \geq c \text{ for all } t \in [0, T] \ \mathbb{P}\text{-a.s.}$$

Remark

The notion of admissibility is not needed in discrete-time models over a finite time horizon because there a doubling strategy has only finitely many chances to win (and thus will lose with positive probability); for infinite horizon models this changes also in discrete-time (see exercise last term).

The admissibility constraint is sometimes difficult to deal with technically—but obviously can't be avoided (just altered at times).

A financially relevant strict local martingale

Obviously, the PnL from the "doubling strategy" constructed above can be written as a stochastic integral

$$V_t = 0 + \int_0^t \xi_s dX_s, \quad t \in [0, T].$$

In particular, when X is a local martingale, also V will be. But because

$$V_0 = 0 \text{ and } V_T = 1 \text{ a.s.}$$

this local martingale cannot be a true martingale; it thus constitutes a financially most relevant example of what is known as a *strict local martingale*.

Remark

Checking whether a local martingale is actually a true martingale can be tedious and technical at times—exactly the kind of stuff mathematicians can get carried away by...

Outline

No arbitrage opportunities among admissible strategies

Lemma

Let $(X_t)_{t\in[0,T]}$ *be a continuous semimartingale on* $(\Omega, (\mathscr{F}_t), \mathbb{P})$ *and suppose that* X *is a local martingale under some* $\mathbb{P}^* \approx \mathbb{P}$ *on* $\mathscr{F}_T$*. Then there is no arbitrage among admissible strategies:*

$$\int_0^T \xi_s dX_s \geq 0 \text{ a.s. and } \xi \text{ admissible} \Rightarrow \int_0^T \xi_s dX_s = 0 \text{ a.s.}$$

Proof

Fundamental theorem of asset pricing

Theorem (Delbaen & Schachermayer (1994))

A locally bounded semimartingale X *on* $(\Omega, (\mathscr{F}_t), \mathbb{P})$ *turns into a local martingale under some* $\mathbb{P}^* \approx \mathbb{P}$ *if and only if* X *satisfies the condition No Free Lunch with Vanishing Risk (NFLVR)*

$$L^\infty\text{-closure}\{g \in L^\infty : g \leq \int_0^T \xi_s dX_s \text{ for some admissible } \xi \in L(X)\} \cap L^\infty_+ = \{0\}.$$

Remark

$f \in L^\infty_+$ is a free lunch with vanishing risk if $f \not\equiv 0$ and if for $n = 1, 2, \ldots$, the initial capital $1/n$ is sufficient to superreplicate it: $0 \leq f \leq \frac{1}{n} + \int_0^T \xi^n_s dX_s$; in other words, the downside risk of the strategies ξ^n is uniformly small, while the upside $\mathbb{P}[f > 0] > 0$ is maintained.

Fundamental theorem of asset pricing

Remark

- This is the best possible result: counter examples.
- Local boundedness can be dispensed with when one relaxes to so-called σ-martingale measures; see Delbaen-Schachermayer 1998.
- The importance of the result lies in deducing rich mathematical structure (local martingale) from general, readily interpretable and acceptable (!) economic conditions.
- The very functional analytic proof would take more than a term to discuss; it seeks to use (NFLVR) in lieu of the martingale property to study upcrossings etc.
- What is important for us is knowing that we should always be able to find a martingale measure if we are looking at a meaningful financial model. Fortunately, in concrete situations these measures are often easy to find.

How to find a martingale measure . . .

? *How to find, for a given continuous semimartingale $X = M + A$, a local martingale measure $\mathbb{P}^* \approx \mathbb{P}$ (if there is one)?*

Recall: For $\mathbb{P}^* \approx \mathbb{P}$ with density process $Z_t := \frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathscr{F}_t}$, $t \in [0, T]$, we have that

$$\begin{aligned} & X \text{ is a (local) } \mathbb{P}^*\text{-martingale} \\ \Longleftrightarrow \quad & XZ \text{ is a (local) } \mathbb{P}\text{-martingale} \end{aligned}$$

Assuming Z continuous too, we thus want

$$d\text{local } \mathbb{P}\text{-martingale} \overset{!}{=} d(XZ) =$$

i.e., with $Z =: \mathscr{E}(L)$:

$$dA \overset{!}{=} -\frac{1}{Z} d[X, Z] = -d[X, L]$$

Girsanov's theorem

Theorem (*Girsanov*)

Let $\mathbb{P}^* \approx \mathbb{P}$ *have continuous density process* $Z = \mathscr{E}(L) = \frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathscr{F}_\cdot}$. *Then*

(i) M^* *is a continuous local martingale under* $\mathbb{P}^*$ *if and only if* $M := M^* + [M^*, L]$ *is a continuous local martingale under* $\mathbb{P}$.

(ii) M *is a continuous local martingale under* $\mathbb{P}$ *if and only if* $M^* := M - [M, L]$ *is a continuous local martingale under* $\mathbb{P}^*$.

Proof

Implications for semimartingales and their integrals

Corollary

(i) *The space of (continuous) semimartingales is the same for equivalent measures.*

(ii) *The space* $L(X)$ *of integrands for a continuous semimartingale* X *is the same for equivalent measures and the stochastic integrals are the same as well.*

$\rightsquigarrow$ PnLs will not change when we consider them under different, but equivalent measures.

Proof

Back to our quest for an equivalent local martingale measure

Necessary for continuous density process Z for $X = M + A$: There is a continuous local martingale $L(= \int_0^\cdot dZ/Z)$ with

$$dA \stackrel{!}{=} -d[X, L].$$

Sufficient: L as above yields via $Z := \mathscr{E}(L)$ not only a continuous *local* martingale (which it always will!?!), but actually a true martingale. Indeed, we can then define a probability measure $\mathbb{P}^* \approx \mathbb{P}$ consistently by putting

$$\mathbb{P}^*[A] := \mathbb{E}[Z_t 1_A] \text{ for } A \in \mathscr{F}_t, \quad t \in [0, T],$$

and this measure will indeed be an equivalent local martingale measure for X.

? *How to tell when the exponential $Z := \mathscr{E}(L)$ of a local martingale L yields a true martingale?*

Nonnegative local martingales

Lemma

Any local martingale M which is uniformly bounded from below is a super-martingale. It is a uniformly integrable true martingale iff $\mathbb{E}[M_\infty] = M_0$.

Example

For a three-dimensional Brownian motion $W = (W^1, W^2, W^3)$ starting at $x \neq 0$, the inverse distance from the origin $(1/|W_t|)$ is a nonnegative strict local martingale.

Remark

The above example illustrates the need for criteria when a nonnegative local martingale is a true martingale.

Corollary

A bounded local martingale is a true uniformly integrable martingale.

Proof

The Kazamaki condition and the Novikov condition

Theorem

For a continuous local martingale L starting in $L_0 = 0$, each of the following statements implies the next:

(i) *Novikov condition:*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}[L]_\infty\right)\right] < \infty$$

$\Longrightarrow$ (ii) *Kazamaki condition: L is a uniformly integrable martingale with*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}L_\infty\right)\right] < \infty.$$

$\Longrightarrow$ (iii) *$\mathscr{E}(L)$ is a uniformly integrable martingale.*

Proof

Outline

Black-Scholes model with stochastic coefficients

Let us generalize the Black-Scholes model and allow stochastic interest rates

$$r = (r_t) \text{ predictable with } \int_0^T |r_t| dt < \infty \text{ a.s.}$$

and an arbitrary Ito-process (rather than a Brownian motion with drift) to drive the asset's returns:

$$\frac{dX_t}{X_t} = \sigma_t dW_t + (\mu_t - r_t) dt$$

for some $\sigma \in L(W)$ and some predictable $\mu \in L^1(dt)$ a.s.
From our theory of stochastic exponentials we know:

$$\begin{aligned} X_t &= s_0 \mathscr{E} \left(\int_0^\cdot \sigma_s dW_s + \int_0^\cdot (\mu_s - r_s) ds \right)_t \\ &= s_0 \exp\left(\int_0^t \sigma_s dW_s + \int_0^t (\mu_s - r_s - \frac{1}{2}\sigma_s^2) ds \right), \quad t \in [0, T]. \end{aligned}$$

Finding an equivalent martingale measure

From our general considerations, we know that we ought to find a continuous local martingale L with

$$d[X, L]_t + dA_t = 0 \text{ where } A_t = \int_0^t X_s(\mu_s - r_s)ds$$

$$\iff$$

$$\iff$$

? *What local martingale L will have this covariation with W?*

From candidate to official martingale measure

So:

$$Z_t := \mathscr{E}(L)_t := \exp\left(-\int_0^t \vartheta_s dW_s - \frac{1}{2}\int_0^t \vartheta_s^2 ds\right), \quad t \in [0, T],$$

is our candidate for the density process $Z_t = \frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathscr{F}_t}$—but we still need to show that this local martingale is a true martingale to get a probability measure $\mathbb{P}^* \approx \mathbb{P}$ this way. Fortunately, we have our criteria worked out for that . . .

Martingale measure for the stochastic Black-Scholes model

Corollary

The process $Z = \mathscr{E}(-\int_0^{\cdot} \vartheta dW)$ *yields the density process of an equivalent local martingale measure for the Black-Scholes model with stochastic coefficients if the market price of risk*

$$\vartheta := \frac{\mu - r}{\sigma} \in L(W)$$

satisfies, e.g., the Novikov-condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\vartheta_s|^2 ds\right)\right] < \infty.$$

Remark

Notice how the Novikov-condition illustrates that the risk (as represented by σ*) has to be in line with returns (as represented by* $\mu - r$*) to rule out arbitrage or even a free lunch with vanishing risk.*

Illustrating Girsanov's theorem for Brownian motion

Theorem

Let W *be a Brownian motion on* $(\Omega, (\mathscr{F}_t), \mathbb{P})$ *and assume* $\mathbb{P}^* \approx \mathbb{P}$ *has a density process of the form*

$$Z_t := \left. \frac{d\mathbb{P}^*}{d\mathbb{P}} \right|_{\mathscr{F}_t} = \exp\left(- \int_0^t \vartheta_s dW_s - \frac{1}{2} \int_0^t \vartheta_s^2 ds \right).$$

Then

$$W_t^* := W_t + \int_0^t \vartheta_s ds$$

is a Brownian motion on $(\Omega, (\mathscr{F}_t), \mathbb{P}^*)$*, respectively*

$$W_t = W_t^* - \int_0^t \vartheta_s ds$$

is a Brownian motion with drift $-\vartheta$ *on* $(\Omega, (\mathscr{F}_t), \mathbb{P}^*)$*.*

Proof

Consequences for Black-Scholes

Corollary

For the stochastic Black-Scholes model with drift μ, interest rate r and volatility σ, we can write the asset price dynamics as

$$\frac{dS_t}{S_t} = \sigma_t dW_t + \mu_t dt = \sigma_t dW_t^* + r_t dt$$

and so, under $\mathbb{P}^*$, the stock price will have the interest rate as its drift.
Equivalently, the discounted stock price evolves according to

$$\frac{dX_t}{X_t} = \sigma_t dW_t + (\mu_t - r_t)dt = \sigma_t dW_t^*,$$

making obvious its local martingale dynamics under $\mathbb{P}^*$.

Lévy's characterization of Brownian motion

Theorem

If M with $M_0 = 0$ is a continuous local martingale on $(\Omega, (\mathscr{F}_t), \mathbb{P})$ with $[M]_t = t$, then it is a Brownian motion on this space.

Remark

This theorem is extremely helpful because establishing that a process is a local martingale is often easy (by virtue of Ito's formula for instance)—and computing quadratic variations is even easier (typically at least, by virtue of our stochastic integration theory and, yet again, Ito's formula).

Proof

Stock price models as time-changed Brownian motions

Theorem (Dambis-Dubins-Schwarz)

Let M be a continuous local martingale with $\lim_{t\uparrow\infty}[M]_t = \infty$. Then there is a standard Brownian motion (in its own filtration) B such that

$$M_t - M_0 = B_{[M]_t}, \quad t \geq 0.$$

Remark

Since we know that the asset prices in every continuous financial model with (NFLVR) can be turned into local martingales, the above theorem implies that *all* reasonable asset price models emerge as time-changed Brownian motions. In that sense, we find that Brownian motion is *the* financial model per se and in some sense the "only" continuous local martingale.

Sketch of proof

Outline

Outline

Arbitrage-free valuations of European claims

Consider a financial model with continuous discounted risky asset price process X admitting $\mathbb{P}^*$ as an equivalent local martingale measure.

? *What is an aribtrage-free valuation for a contingent claim with discounted payoff $H \geq 0$ at time T?*

! As in discrete-time models, any number π for which we can find an "interpolating" price process X^H which starts in $X_0^H = \pi$ and ends in $X_T^H = H$ and which keeps the model extended by X^H arbitrage-free.

If H is $\mathbb{P}^*$-integrable such an interpolation is readily found for $\pi := \mathbb{E}^*[H]$, namely

$$X_t^H := \mathbb{E}^* \left[H | \mathscr{F}_t \right], \quad t \in [0, T].$$

Indeed: Then $\mathbb{P}^*$ will be a martingale measure for both X and X^H and, thus, ensure absence of arbitrage in the extended model by the FTAP.

Illustration in the Black-Scholes model: power options

In the Black-Scholes model with constant coefficients, consider

$$H = e^{-rT} S_T^p \text{ — a ``power option''}.$$

Under $\mathbb{P}^* \approx \mathbb{P}$ with density process $\mathscr{E}(-\frac{\mu-r}{\sigma}W)$, this H is integrable and so an arbitrage free price process is

$$\begin{aligned}
X_t^H &= \mathbb{E}^*\left[H|\mathscr{F}_t\right] = e^{-rT}\mathbb{E}^*\left[s_0^p \exp\left(p\sigma W_T^* + p(r - \frac{1}{2}\sigma^2)T\right)\middle|\mathscr{F}_t\right] \\
&= e^{-rT} s_0^p \exp\left(p\sigma W_t^* + \frac{1}{2}p^2\sigma^2(T-t) + p(r - \frac{1}{2}\sigma^2)T\right) \\
&= X_t^p \exp\left(\frac{1}{2}(p - p^2)\sigma^2 t + (\frac{1}{2}(p^2 - p)\sigma^2 + (p-1)r)T\right) \\
&= X_t^p \exp\left(\frac{1}{2}(p^2 - p)\sigma^2(T-t) + (p-1)rT\right) \\
&= X_t^p \exp\left((p-1)\left(\frac{1}{2}p\sigma^2(T-t) + rT\right)\right)
\end{aligned}$$

Alternative arbitrage free prices?

The construction of our equivalent martingale measure $\mathbb{P}^*$ did not rule out the possibility of other martingale measures.

? *Would other martingale measures lead to different prices?*

! Not if the power option is replicable!

So let us check:

$$\begin{aligned}\xi_t^H dX_t &\overset{?}{=} dX_t^H \\ &= d\left(X_t^p \exp\left((p-1)\left(\frac{1}{2}p\sigma^2(T-t)+rT\right)\right)\right) \\ &= \underbrace{pX_t^{p-1}\exp(\dots)}_{=:\xi_t^H} dX_t + \underbrace{0}_{\text{because } X^H \text{ is a } \mathbb{P}^*\text{-martingale!}} dt\end{aligned}$$

Hence, ξ^H is replicating the power option—and it is admissible because its gains process is $G(\xi^H) = X^H - X_0^H \geq -X_0^H$.

Note: Obviously hedging strategies have to be admissible, too!

Trouble on the horizon: More expensive replication!

Caveat

The negative of our "doubling strategy" from above is an admissible strategy starting in 0 and ending a.s. in -1 ("suicide strategy"). If we add it to the above replicating strategy, it will replicate the option starting with initial wealth $\mathbb{E}^*[H] + 1$—and in fact also this is an arbitrage-free price for the power option!

- **?** *How can there be two arbitrage-free prices and replication?*
- **!** To take advantage of the higher arbitrage-free price we would want to replicate the option using the lower price as initial capital and run the negative of the "other" replication strategy—but the net strategy then is the "doubling strategy" which is inadmissible!!

$\leadsto$ replication alone (even by admissible strategies) does no longer suffice to pin down good arbitrage-free prices

Good news: pricing by expectation cheapest

Fortunately, our recipe of pricing and, if possible, replication is guaranteed to be the most competitive approach:

Lemma

If $H \geq 0$ is $\mathscr{F}_T$-measurable and integrable with respect to an equivalent local martingale measure $\mathbb{P}^*$ for X such that

$$X_t^H := \mathbb{E}^*[H|\mathscr{F}_t] = \pi^H + \int_0^t \xi_s^H dX_s, \quad t \in [0, T],$$

for some $\pi^H \in \mathbb{R}$, $\xi^H \in L(X)$, then π^H is minimal among all $\pi \in \mathbb{R}$ for which there is an admissible $\xi \in L(X)$ such that

$$\pi + \int_0^T \xi_s dX_s \geq H \text{ a.s.}$$

Moreover, X^H and, in particular, $\pi^H = X_0^H$ do not depend on the choice of martingale measure $\mathbb{P}^*$ if such a replicating ξ^H exists.

Proof

Did we get lucky with our replication of the power option?

On the one hand: No - we already know since FiMa I that vanilla options in the Black-Scholes model can be replicated (and now we know that our recipe will lead to admissible strategies in a natural way).

On the other hand: Yes - for, what if the option price dynamics dX_t^H cannot be expressed in terms of the underlying's dX_t alone?

Unspanned risk

Example

Let S be as in the Black-Scholes model and for an independent (for simplicity) Brownian motion B take $E = \mathscr{E}(\sigma_E B)$ as the model for an exchange rate (say). Consider a power option on a foreign stock denominated in domestic currency:

$$H' := e^{-rT} S_T^p E_T$$

Then, using $\mathbb{P}^*$ from above, we find

$$X_t^{H'} = \mathbb{E}^*\left[e^{-rT} S_T^p E_T \middle| \mathscr{F}_t\right] \overset{?!}{=} X_t^H E_t$$
$$dX_t^{H'} = E_t dX_t^H + X_t^H dE_t = E_t \xi_t^H X_t \sigma dW_t^* + X_t^H E_t dB_t$$

So hedging with the stock X will not suffice for replication, we also need to hedge the exchange rate risk—and would require an extra asset for that.

Outline

Hedging when you cannot replicate

Consider a financial model with

- $X = (X_t)$: continuous discounted asset price for a stock
- $\mathbb{P}^*$: an equivalent measure such that $X \in \mathscr{M}_c^2(\mathbb{P}^*)$
- $H \in L^2(\mathbb{P}^*)$: a discounted contingent claim that we need to cover at time T

? *If we cannot replicate H by trading in X, how should we hedge against this liability?*

! Conservative answer: Superreplicate it!

? *But what if this is too costly—after all we cannot, in good faith, charge our clients the super-replication price?!?*

! Many alternative answers, one of them:
Quadratic hedging à la Föllmer-Sondermann:

$$\text{Minimize } \mathbb{E}^*\left[\left(H - (v_0 + \int_0^T \xi_s dX_s)\right)^2\right]$$

over pred. $\xi \in L^2(\mathbb{P}^* \otimes d[X])$ and, possibly, also over $v_0 \in \mathbb{R}$.

Decomposition square-integrable martingales

Theorem (*Kunita-Watanabe decomposition*)

For $(X_t) \in \mathscr{M}^2(\mathbb{P})$, every square-integrable $\mathbb{P}$-martingale $(H_t) \in \mathscr{M}^2(\mathbb{P})$ has a unique decomposition

$$H_t = H_0 + \int_0^t \xi_s^H dX_s + N_t, \quad t \in [0, T],$$

where

- ξ^H *is predictable with* $\mathbb{E}^* \left[\int_0^T (\xi_s^H)^2 d[X]_s \right] < \infty$, *i.e.,* $\xi^H \in L^2(X)$,
- N *is a square-integrable* $\mathbb{P}$*-martingale which is strongly orthogonal to* X *in the sense that* $\mathbb{E}[X_S N_S] = 0$ *for any stopping time* $S \leq T$ *(i.e., such that* XN *is a martingale),*
- $H_0 = \mathbb{E}^*[H_T]$.

Proof

Quadratic hedging solved

Corollary

Put $H_t := X_t^H := \mathbb{E}^*[H|\mathscr{F}_t]$, $t \in [0, T]$, *and consider its Kunita-Watanabe-decomposition*

$$H_t = H_0 + \int_0^t \xi_s^H dX_s + N_t, \quad t \in [0, T].$$

Then ξ^H *(and, if we also want to maximize over the initial captial,* $v_0^H := H_0 = X_0^H$*) minimize*

$$\mathbb{E}^* \left[\left(H - (v_0 + \int_0^T \xi_s dX_s) \right)^2 \right]$$

over predictable $\xi \in L^2(\mathbb{P}^* \otimes d[X])$ *(and, possibly, also over* $v_0 \in \mathbb{R}$*).*

Proof

Discussion

- $v_0^H = H_0 = X_0^H = \mathbb{E}^*[H]$ obviously depends on our choice of martingale measure $\mathbb{P}^*$ for X, as does the strategy ξ^H.
- Taking v_0 as a price for H is an arbitrage-free valuation, but it leaves all risk related to the "unhedgeable" strongly orthogonal component N unpriced.
- The $L^2(\mathbb{P}^*)$-distance is obviously mathematically convenient, but economically not beyond reproach. For instance, it penalizes losses the same it penalizes gains.

? *So wouldn't it be good to have a model where everything is replicable after all?*

Outline

Ito's representation theorem

Theorem

For a Brownian motion W on $(\Omega, (\mathscr{F}_t), \mathbb{P})$, we have:

(i) *Every local martingale M which is adapted to the filtration $(\mathscr{F}_t^W)$ generated by W is of the form*

$$M = M_0 + \int_0^{\cdot} \xi_s dW_s \text{ for some } \xi \in L(W).$$

In particular, every "Brownian" local martingale is continuous.

(ii) *For any $T \geq 0$, every $\mathscr{F}_T^W$-measurable square-integrable functional H of the Brownian motion W is of the form*

$$H = \mathbb{E}[H] + \int_0^T \xi_s dW_s \text{ for some } \xi \in L^2(\Omega \times [0, T], \mathscr{P}, \mathbb{P} \otimes dt).$$

(iii) *The integrand ξ in the Ito-representations of (i) and (ii) is unique in its respective class.*

Counterexample for uniqueness in (ii) within $L(W)$ instead of $L^2(W)$

Take ξ as in (ii) and let $\xi^d \in L(W)$ be a doubling strategy over $[0, T/2]$ such that $\int_0^{T/2} \xi^d dW = 1$, $\xi^d = 0$ on $[T, T/2]$, and let $\xi^s \in L(W)$ be a suicide strategy over $[T/2, T]$ such that $\xi^s = 0$ on $[0, T/2]$ and $\int_{T/2}^T \xi^s dW = -1$.

Then $\tilde{\xi} := \xi + \xi^d + \xi^s \in L(W)$ also represents

$$\mathbb{E}[H] + \int_0^T \tilde{\xi} dW = \mathbb{E}[H] + \int_0^T \xi dW + 1 - 1 = H$$

but obviously differs from ξ. Since obviously (?!) $\tilde{\xi} \notin L^2(W) = L^2(\Omega \times [0, T], \mathscr{P}, \mathbb{P} \otimes dt)$, this is a counterexample to uniqueness in (ii) within $L(W)$, but is contradiction to the uniqueness in $L^2(W)$ stated in item (ii) of Ito's representation theorem.

Proof

Completeness of the Black-Scholes model

Corollary

With respect to the filtration $\mathscr{F}_t := \mathscr{F}_t^S = \mathscr{F}_t^W = \mathscr{F}_t^X$, $t \in [0, T]$, *our Black-Scholes model with constant coefficients driven by the Brownian motion* W *is complete in the sense that for any bounded contingent claim* $H \in \mathscr{F}_T^S$ *there is an initial capital* $v^H \in \mathbb{R}$ *and a strategy* $\xi^H \in L(X)$ *with bounded wealth process such that*

$$H = v^H + \int_0^T \xi_s^H dX_s.$$

Proof

Uniqueness of martingale measure in Black-Scholes model

Corollary

In the Black-Scholes-model with constant coefficients and filtration $\mathscr{F}_t := \mathscr{F}_t^S = \mathscr{F}_t^W$, $t \geq 0$, *the measure* $\mathbb{P}^* \approx \mathbb{P}$ *on each* $\mathscr{F}_t$ *with density process*

$$Z_t = \left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathscr{F}_t} := \mathscr{E}(-\vartheta W)_t = \exp\left(-\vartheta W_t - \frac{1}{2}\vartheta^2 t\right) \text{ for } \vartheta := \frac{\mu - r}{\sigma}$$

is the only equivalent local martingale measure for the discounted stock price $X_t = e^{-rt} S_t$, $t \geq 0$.

Proof

Outline

Completeness in general

Corollary

For a continuous discounted stock price process $X = (X_t)_{t\in[0,T]}$ *on* $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$, *the following two statements are equivalent:*

(i) *There exists a unique equivalent local martingale measure* $\mathbb{P}^*$ *for* X *on* $\mathscr{F}_T$.

(ii) X *satisfies (NFLVR) and the financial model is complete in the sense that any bounded contingent claim is replicable, i.e., for any bounded* $H \in \mathscr{F}_T$ *there is an initial capital* $v^H \in \mathbb{R}$ *and a strategy* $\xi^H \in L(X)$ *with bounded wealth process* V^{v^H,ξ^H} *such that*

$$H = V_T^{v^H,\xi^H} = v^H + \int_0^T \xi_s^H dX_s.$$

Proof

Attainable claims

Corollary

Let X be continuous and satisfy (NFLVR) on $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$. Then the following statements are equivalent for any bounded claim $H \in \mathscr{F}_T$:

(i) *H is replicable (or attainable), i.e., $H = V_T^{v^H,\xi^H}$ for some $v^H \in \mathbb{R}$ and some $\xi^H \in L(X)$ with bounded wealth process V^{v^H,ξ^H}.*

(ii) *$\mathbb{E}^*[H]$ does not depend on the choice of equivalent martingale $\mathbb{P}^* \approx \mathbb{P}$ for X on $\mathscr{F}_T$.*

Proof

Discussion

- The above results also holds for multivariate models where more than one asset is available for trading.
- The characterizations of completeness and attainability are true in models with jumps as well, but much harder to prove there.

The hunt is on . . .

The previous results give more than enough reasons to

- specify financial models with equivalent martingale measures
- determine derivative prices by (the comparably easy task of) computing expectations
- and use the latter to find hedging strategies
- see how they perform in “the market” (i.e., in reality)
- improve and refine the models where and when they fall short

Part II

Financial modelling

Disclaimer

The focus in this part of our course will be on conceptual aspects of financial models, not necessarily their mathematically completely rigorous analysis.

So: We will not distinguish between local martingales and true martingales; processes will be as integrable as required implicitly; functions will be smooth enough for applying Ito's formula whenever we want to...

Don't worry: You will still learn a lot of very useful mathematics!

Outline

Outline

A Brownian framework

Standing assumption:

The information flow is the natural filtration $(\mathscr{F}_t^B)$ generated by a d-dimensional Brownian motion $B = (B^1, \ldots, B^d)$ on $(\Omega, \mathscr{F}, \mathbb{P})$.

- market driven by random shocks from B: risk factors
- interest rates (r_t) almost surely locally dt-integrable, $(\mathscr{F}_t^B)$-predictable
- stock price process (S_t) is $(\mathscr{F}_t^B)$-adapted, right-continuous

Structure theorem in a Brownian framework

Theorem

Every stock price model (S_t) *that exhibits (NFLVR) on* $(\Omega, \mathscr{F}, (\mathscr{F}_t^B), \mathbb{P})$ *has dynamics of the form*

$$\begin{aligned} \frac{dS_t}{S_t} &= (r_t + \vartheta_t \cdot \sigma_t^B)dt + \sigma_t^B \cdot dB_t \\ &= \mu_t dt + \sigma_t dW_t, \quad t \geq 0, \end{aligned}$$

for some predictable $\vartheta, \sigma^B \in L(W)$*, and for some 1-dimensional Brownian motion* W*, some* $\sigma \in L(W)$ *and some predictable* $\mu \in L^1_{\mathrm{loc}}(dt)$ *almost surely.*

Proof

Interpretation of structure theorem

Remark:

- ϑ_t^i: *market price of risk* that the pricing measure $\mathbb{P}^*$ attributes to exposure to the risk shock dB_t^i, $i = 1, \ldots, d$
- $\sigma_t^B := \lambda_t + \vartheta_t$: extent to which the asset S is driven by the shock dB_t
- $\mu_t - r_t = \vartheta_t \cdot \sigma_t^B$: *risk premium* for holding asset S under $\mathbb{P}^*$
- The structure theorem also holds mutatis mutandis in the multivariate case $S = (S^1, \ldots, S^n)$.

The one-dimensional case

Corollary

If $d = 1$ and $\sigma_t^2 = \frac{d[S]_t}{S_t^2} > 0$ $\mathbb{P} \otimes dt$-a.e., then the Brownian financial model is complete. Otherwise, it is incomplete.

Remark

This result generalizes to the multivariate case and, similarly to discrete time, one finds—morally speaking—that a model will be complete if and only if the number d of risk factors coincides with the "true" number of traded assets.

Proof

Outline

Model misspecification and Black-Scholes

Under an arbitrary local martingale measure $\mathbb{P}^* \approx \mathbb{P}$, any arbitrage free Brownian financial model has dynamics of the form

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t^*, \quad t \geq 0,$$

for a suitable $\mathbb{P}^*$-Brownian motion W^*.
Financial modeling thus reduces to specification of

- interest rates (r_t)
- volatility (σ_t)

For now focus on volatility: Assume $r = 0$; see below for interest rate models.

? *How sensitive is our option pricing and hedging theory with respect to misspecification of volatility?*

Labtest: In a Black-Scholes trader's shoes ...

Consider a trader who

- operates in a Brownian market with $r_t \equiv 0$ and **unknown** volatility (σ_t)
- postulates for his risk management computations that **volatility is constant**: $\widehat{\sigma}_t \equiv \widehat{\sigma} \in (0, \infty)$

$\rightsquigarrow$ trader implements model under $\widehat{\mathbb{P}}^*$ with

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = \widehat{\sigma}\, d\widehat{W}_t^*$$

using filtration $\widehat{\mathscr{F}_t} = \mathscr{F}_t^{\widehat{S}}$, $t \geq 0$.

Notice:

Quantities featuring in the trader's model and computer implementation are wearing a hat; their counterparts in "reality" do not.

Labtest: In a Black-Scholes trader's shoes ...

$\rightsquigarrow$ trader prices derivative $H = f(S_T)$ as $\widehat{H} = f(\widehat{S}_T)$ via

$$\pi_t(H)(\omega) =$$

$\rightsquigarrow$ trader computes solution $\widehat{v} = \widehat{v}(t, s)$ to Black-Scholes-PDE

$$\partial_t \widehat{v} + \frac{1}{2}\widehat{\sigma}^2 s^2 \partial_s^2 \widehat{v} = 0, \quad \widehat{v}(T, .) = f$$

$\rightsquigarrow$ trader quotes price $\pi_0(H) = \widehat{v}(0, S_0)$ and runs hedge $\Delta_t = \partial_s \widehat{v}(t, S_t)$, $t \in [0, T]$.

? *How effective is this hedge?*

Over- and underhedging with vol errors

Theorem

Assume the payoff profile f is convex. In a scenario ω with

$$\sigma_t(\omega) \leq \widehat{\sigma} \quad \textit{for} \quad t \in [0, T],$$

the trader's hedge Δ will super-replicate the contingent claim:

$$V_t = \qquad \geq$$

$$V_T \geq$$

Underestimating volatility in a scenario will similarly result in a "sub-replication".

Proof

Outline

Outline

Implied volatility

Recall: Black-Scholes call prices are strictly increasing in volatility parameter from 0 to ∞:

- Vega $= \frac{\partial}{\partial\sigma}$BS-call price$(T, k, s_0, r, \sigma) > 0$
- $(0, \infty) \ni \sigma \mapsto$ BS-call price$(T, k, s_0, r, \sigma) \in \left(\left(s_0 - ke^{-rT}\right)^+, s_0\right)$ bijectiv

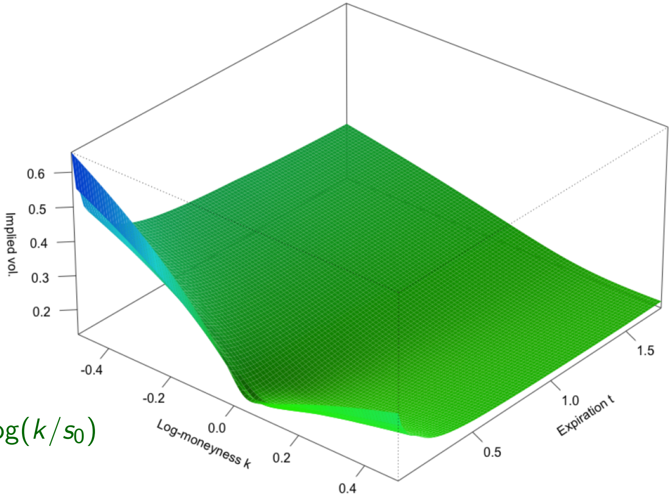
$\rightsquigarrow$ Given strike and maturity, call prices can be encoded by the volatility consistent with these prices: implied volatility.

$$\text{observed call price} \overset{!}{=} \text{BS-call price}(T, k, s_0, r, \sigma_{\text{imp}})$$

$\rightsquigarrow$ Given today's stock price s_0 and interest r, $(T, k) \mapsto \sigma_{\text{imp}}(T, k, s_0, r)$ is called (implied) volatility surface

? *Is that really an interesting object to consider?*

Implied volatility surface



$\kappa = \log(k/s_0)$

Figure: S&P implied volatility surface as of June 20, 2013.
Source: Gatheral, Jaisson, Rosenbaum, Volatility is rough, 2014

Why implied volatility?

Fun fact:

Traders typically quote call prices in terms of implied vol rather than actual prices because implied volatility accounts for moneyness and maturity and makes option price quotes more easily comparable!

? *How to interpret?*

!
- Option prices (translated into implied vol quotes) further "in" ($\log(k/s_0) < 0$) or "out of the money" ($\log(k/s_0) > 0$) are higher than one would anticipate from those "at the money" ($\log(k/s_0) = 0$).
- Observed first after crash of October 19, 1987 ("black monday"): -22% loss in Dow
- Constant vol insufficient to capture such extreme market dynamics

Black Monday and intraday volatility

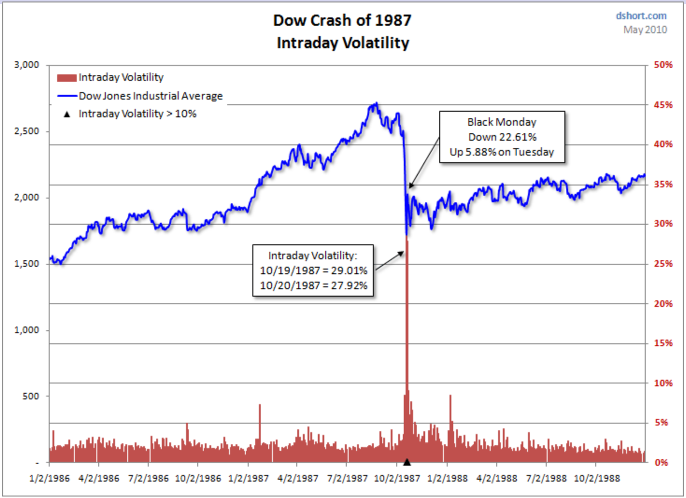


Figure: Dow crash on October 19, 1987 and intraday volatility.

Source: https://seekingalpha.com/article/204107-eight-decades-of-market-volatility

A guided tour through volatility models

Hull & White model (1987)

$$\frac{dS_t}{S_t} =$$

A guided tour through volatility models

Scott model (1987)

$$\frac{dS_t}{S_t} =$$

A guided tour through volatility models

Stein & Stein model (1991)

$$\frac{dS_t}{S_t} =$$

A guided tour through volatility models

Heston model (1993)

$$\frac{dS_t}{S_t} =$$

A guided tour through volatility models

Dupire's local vol model (1994)

$$\frac{dS_t}{S_t} =$$

A guided tour through volatility models

SABR model (Hagan 2003)

$$\frac{dS_t}{S_t^{\beta}} =$$

$$=$$

Outline

More on Dupire's 'local vol'-model

? *How to determine the volatility profile $\widehat{\sigma} = \widehat{\sigma}(t, s)$ in Dupire's model*

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = \widehat{\sigma}\left(t, \widehat{S}_t\right) d\widehat{W}_t^*, \widehat{S}_0 = S_0?$$

! By making sure that vanilla options are priced correctly: **model calibration**

$$\text{market price for } H = f(S_T) \stackrel{!}{=} \text{model price for } \widehat{H} = f(\widehat{S}_T)$$
$$= \widehat{\mathbb{E}}^*[f(\widehat{S}_T)] = \widehat{v}(0, S_0)$$

for *all* payoff profiles f and *all* maturities $T > 0$.

Calibration of local vol

Assumption:

The market prices risks according to $\mathbb{P}^*$ under which the asset price evolves according to

$$\frac{dS_t}{S_t} = \sigma_t dW_t^*$$

for some unknown "real-world" volatility process σ.

Then calibration of Dupire's 'local vol'-model to vanilla option prices amounts to finding $\widehat{\sigma} = \widehat{\sigma}(t, s)$ such that

$$\mathbb{E}^*[f(S_T)] \stackrel{!}{=} \widehat{\mathbb{E}}^*[f(\widehat{S}_T)] \text{ for all payoff profiles } f \text{ and all } T > 0.$$

In other words: The model $\widehat{S}$ exhibits the same 1-dimensional marginal distributions as the market assumes for S.

Calibration of local vol

A mimicking theorem

Theorem (Gyöngy 1987, ...)

Let S have volatility σ under $\mathbb{P}^*$ and suppose

$$\widehat{\sigma}(t, s) := \left(\mathbb{E}^* \left[\sigma_t^2 \middle| S_t = s\right]\right)^{1/2}$$

is sufficiently regular. Then the solution $\widehat{S}$ of the SDE

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = \widehat{\sigma}\left(t, \widehat{S}_t\right) d\widehat{W}_t^*$$

has the same 1-dimensional marginal distributions as S:

$$\mathrm{Law}\left(S_t | \mathbb{P}^*\right) = \mathrm{Law}(\widehat{S}_t | \widehat{\mathbb{P}}^*), \quad t \in [0, T].$$

Dupire's formula

? *How to determine the expectations* $\mathbb{E}^*\left[\sigma_t^2 \middle| S_t = s\right]$, $s > 0$, *from market data?*

Assumption:

Call option prices for arbitrary strikes k and maturities T are observable, i.e., the mapping

$$c : (T, k) \mapsto c(T, k) = \mathbb{E}^*[(S_T - k)^+]$$

is given.

Remark

Only approximately true in practice, but a good starting point for theoretical investigations.

Dupire's formula

Observation 1:

Call option prices c determine the 1-dimensional marginals of both S under $\mathbb{P}^*$ and $\widehat{S}$ under $\widehat{\mathbb{P}}^*$.

Dupire's formula

Observation 2:

Any choice of $\widehat{\sigma} = \widehat{\sigma}(t, s)$ also yields 1-dimensional marginals for $\widehat{S}$ under $\widehat{\mathbb{P}}^*$ as determined by the *Fokker-Planck-equation*.

Dupire's formula

Discussion of Dupire's formula

$$\widehat{\sigma}^2(T,k) = \frac{2}{k^2}\frac{\partial_T c(T,k)}{\partial_k^2 c(T,k)}$$

- in reality only finitely many call option prices given and only those at the money trading with sufficient liquidity to reliably reflect market view

$\rightsquigarrow$ need for interpolation schemes

- But how to ensure that such a scheme does not lead to arbitrage?

Fact/reminder:

A call option price mapping c does not allow for arbitrage if and only if

(i) $k \mapsto c(T,k)$ is decreasing and convex with $c(T,0+) = S_0$ and $\lim_{k\uparrow\infty} c(T,k) = 0$.

(ii) $T \mapsto c(T,k)$ is increasing.

Discussion of Dupire's formula

Conceptual problem: Intrinsic lack of consistency in time because model is static!

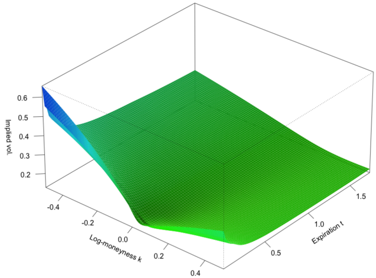


Figure: S&P implied volatility surface as of June 20, 2013.
Source: Gatheral, Jaisson, Rosenbaum, Volatility is rough, 2014

Built-in need for constant recalibration. Not really a model, certainly not a good one.
$\rightsquigarrow$ Need models that are dynamically consistent and that can still be calibrated, at least to some extent.

Outline

More on the Heston model

$$\frac{d\hat{S}_t}{\hat{S}_t} = \sqrt{V_t}\,d\hat{W}_t^*$$
$$dV_t = \alpha(\beta - V_t)dt + \gamma\sqrt{V_t}\,dB_t$$

Existence of a nonnegative, strong solution to the latter SDE is a classical result due to Yamada & Watanabe (see Karatzas-Shreve):

$$V_t = \mathscr{V}(v_0, B)_t \text{ for some } \mathscr{V} : \mathbb{R}_+ \times C[0,\infty) \to C[0,\infty)$$

? *Does volatility ever vanish?*

Feller test for explosions in the Heston model

Theorem

If $d := 4\alpha\beta/\gamma^2 < 2$, we have $\hat{\mathbb{P}}^*[V_t = 0$ for some $t \geq 0] = 1$; if $d \geq 2$ we have $\hat{\mathbb{P}}^*[V_t = 0$ for some $t \geq 0] = 0$.

Interpretation

So if the upward drift $\alpha\beta$ (effective whenever V is small) is large enough compared to the local variance $\gamma^2 V$ in that regime, level 0 will not be reached — otherwise it will be.
While intuitive, the precision with which these cases can be distinguished is due to Feller's test for explosions which allows one to study the behavior of one-dimensional diffusions concerning the boundary of their domains.

Proof of Feller test for explosions in the Heston model

Distributional properties of the Heston model

Theorem

If $d := 4\alpha\beta/\gamma^2 \in \{2, 3, 4, \dots\}$ *then*

$$(V_t)_{t\geq 0} \overset{(law)}{=} (\sum_{i=1}^{d} (X_t^i)^2)_{t\geq 0}$$

for d *independent Ornstein-Uhlenbeck processes*

$$dX_t^i = \frac{\gamma}{2} dW_t^i - \frac{\alpha}{2} X_t^i dt$$

with starting points such that $\sum_{i=1}^{d} (X_0^i)^2 = v_0$.

Hence, $\sqrt{V}$ is the radial part of a d-dimensional Ornstein-Uhlenbeck process. That V never reaches zero for $d \geq 2$ is thus easy to understand.

Distributional properties of the Heston model

Distributional properties of the Heston model

Theorem (Dragulescu, Yakovenko, 2002)

Given starting values (s_0, v_0), $\log(\hat{S}_t/S_0)$ *has the density*

$$f_t(s) = \frac{1}{\pi}\int_0^\infty \mathrm{Real}(\exp(izs + F_t(z)))dz$$

where

$$\begin{aligned} F_t(z) = & - v_0 \frac{z^2 - iz}{c + \lambda \coth(\lambda t/2)} + \frac{\alpha\beta c t}{2} \\ & - \frac{2\alpha\beta}{\gamma^2} \log\left(\cosh(\lambda t/2) + \frac{c}{\lambda}\sinh(\lambda t/2)\right) \end{aligned}$$

with $c = \alpha + i\rho\gamma t$, $\lambda = \sqrt{cz^2 + \gamma^2(z^2 - iz)}$.

Distributional properties of the Heston model

Theorem (Heston 1993)

Call-option prices in the Heston model are given by

$$\hat{\mathbb{E}}^*[(\hat{S}_T - k)^+] = S_0 P_1(s_0, v_0, T) - k P_2(s_0, v_0, T)$$

with

$$P_j(s, v, T) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Real} \frac{\exp(izlogk)\hat{P}_j(s, v, T, z)}{iz} dz$$

where

$$\hat{P}_j(s, v, T, z) = \exp(A_j(T, z) + B_j(T, z)v + iz \log(s))$$

Distributional properties of the Heston model

Theorem (Heston 1993 (continued))

for

$$A_j(T,z) = \frac{a}{\gamma^2}((b_j - \rho\gamma z i + d_j)T - 2\log\frac{1 - g_j e^{d_j T}}{1 - g_j})$$

$$B_j(T,z) = \frac{b_j - \rho\gamma z i + d_j}{\gamma^2}\frac{1 - g_j e^{d_j T}}{1 - g_j}$$

and the constants

$$a = \alpha\beta, b_1 = \alpha - \rho\gamma, b_2 = \alpha, \mu_1 = 1/2, \mu_2 = -1/2,$$

$$d_j = \sqrt{(\rho\gamma z i - b_j)^2 - \gamma^2(2\mu_j z i - z^2)},$$

$$g_j = \frac{b_j - \rho\gamma z i + d_j}{b_j - \rho\gamma z i - d_j}.$$

Upshot on the Heston model

- The affine structure of the Heston model allows for computationally highly efficient Fourier-techniques to do option pricing. (More on this in the special lecture *Computational Finance*; exercise.)
- Fast calibration of the model parameters becomes possible this way. See `https://demonstrations.wolfram.com/VolatilitySurfaceInTheHestonModel/`

Outline

A brief look at recent models

Rough volatility model (Gatheral et al. 2014, ...)

$$\frac{d\hat{S}_t}{\hat{S}_t} = \hat{\sigma}_t d\hat{W}_t^*$$

where in Gatheral et al. $\hat{\sigma}_t = \exp(X_t)$ for X with dynamics

$$dX_t = \alpha(\beta - X_t)dt + \gamma dB_t^H$$

where $H \in (0,1)$ is Hurst parameter of fractional Brownian motion

$$B_t^H = \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}} \left(\int_{-\infty}^{t} \frac{dB_s}{(t-s)^{1/2-H}} - \int_{-\infty}^{0} \frac{dB_s}{(t-s)^{1/2-H}} \right)$$

Gaussian process with correlation function

$$\hat{\mathbb{E}}^*[B_s^H B_t^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

A brief look at recent models: Rough volatility

- Excellent fit for at-the-money volatility skew

$$\psi(T) = \left| \frac{\partial}{\partial \kappa}\bigg|_{\kappa=0} \sigma_{\mathrm{imp}}(\kappa, T) \right| \overset{T \downarrow 0}{\approx} \frac{1}{T^{1/2-H}} \quad \text{with } \kappa = \log k/s_0$$

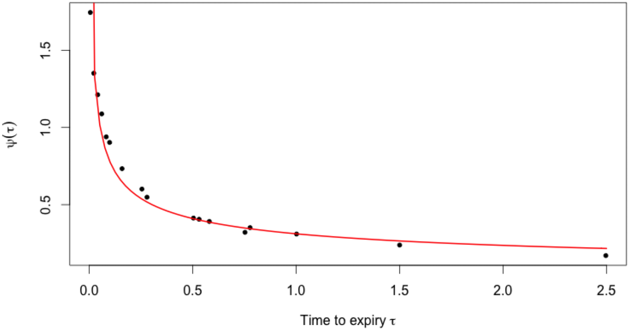


Figure: Estimated S&P skew, June 20, 2013 and power-law fit $\psi(\tau) = A\tau^{-0.4}$.
Source: Gatheral, Jaisson, Rosenbaum, Volatility is rough, 2014

A brief look at recent models: Rough volatility

- Data suggests $H \approx 0.1$, far away from the diffusion like fluctuations (which have Hurst parameter $1/2$)
- Previous models have finite skew for $\tau \downarrow 0$.
- neither Markovian nor a semimartingale unless $H = 1/2$
- model can also be calibrated to forward variance curve $T \mapsto \mathbb{E}^*[\sigma_T^2]$
- theory of rough affine models like rough Heston (Rosenbaum-El Euch '17)

$$\sigma_t = \sqrt{V_t}$$

$$V_t = v_0 + \int_0^t \alpha(\beta - V_u)\frac{du}{\Gamma(1/2+H)(t-u)^{1/2-H}} + \int_0^t \gamma\sqrt{V_u}\frac{dB_u}{\Gamma(1/2+H)(t-u)^{1/2-H}}$$

- microeconomic foundation from order-book dynamics

A brief look at recent models: Local stochastic vol

Local stochastic vol model (Guyon & Henry-Labordère '16)

$$\frac{d\hat{S}_t}{\hat{S}_t} = a_t \hat{\sigma}(t, \hat{S}_t) d\hat{W}_t^*$$

- From Gyöngy's result, we will have the "right" one-dimensional marginals if

$$\hat{\sigma}^2_{\text{Dupire}}(t, \hat{S}_t) = \hat{\mathbb{E}}^*[a_t^2 | \hat{S}_t] \hat{\sigma}^2(t, \hat{S}_t)$$

- free to choose a, e.g. as in Heston
- above choice of $\hat{\sigma}$ leads to novel McKean-Vlasov-dynamics

$$\frac{d\hat{S}_t}{\hat{S}_t} = \frac{a_t}{\sqrt{\hat{\mathbb{E}}^*[a_t^2 | \hat{S}_t]}} \hat{\sigma}_{\text{Dupire}}(t, \hat{S}_t) d\hat{W}_t^*$$

A brief look at recent models: Local stochastic vol

- open problem to prove existence and uniqueness for interesting a
- conditioning: singular dependence of dynamics on $\mathrm{Law}(\hat{S}_t, a_t)$
- but: efficient calibration possible via use of particle methods
- open problem: prove of convergence of particle method under singular McKean-Vlasov dynamics ("propagation of chaos")

Outline

Outline

Bonds

Firms can raise the money to fund themselves in several ways:

- draw on lines of credit from their banks
- issue shares of stock
- issue bonds

States find the money they spend mostly by

- collecting taxes
- issuing bonds

Key difference

- Shares of stock give ownership of the company and exposure to its profits and losses (with limited liability).
- Bonds specify precise future payments (coupons) to be made by the issuer until their maturity when their notional value is due. Exposure to default and inflation risk.

? *How to account for this difference in a financial model?*

Model what is traded: (Zero coupon) bonds

- A zero coupon bond pays its holder 1 unit of currency (notional) at its maturity T (and nothing else).
- building block of more complex bonds
- price process $P_{\cdot}(T) = (P_t(T))_{t \in [0,T]} \geq 0$
- clearly $P_T(T) = 1$, neglecting default, but typically $P_t(T) \neq 1$ stochastic because of changes in time value of money, perception of default risk or inflation, ...
- at any time t same issuer can have bonds that differ in value with maturity: term structure $[t, \infty) \ni T \mapsto P_t(T)$

Challenges:

- Find “reasonable” nonnegative stochastic processes ending up at the deterministic value 1.
- Ensure “reasonable” term structure and dependence of random field $(P_t(T))_{t \geq 0, T \geq t})$.
- Comparing bond prices over wide ranges of maturity may not make sense: accumulation of risk over time

Model interest rates - but which?

Forward rate agreement (FRA)

- What interest can I secure at time t for $1\$$ to be invested "safely" at time $S > t$ until $T > S$?
- simply compounded forward rate for $[S, T]$ contracted at time $t \leq S$
- link to zero bond prices:

$$L_t(S, T) :=$$

i.e.

$$\frac{P_t(S)}{P_t(T)} =$$

Note: The factor $T - S$ is used for normalization by the period length to obtain comparable quantities $L_t(.,.)$.

Model interest rates - but which?

Forward rate agreement (FRA) - no arbitrage argument

Model interest rates - but which?

Simple spot rate for $[t, T]$

- interest rate agreement to start "on the spot": $S = t$
- link to zero bond prices

$$L_t(T) :=$$

i.e.

$$\frac{1}{P_t(T)} =$$

Model interest rates - but which?

Continuously compounded forward rate for $[S, T]$ contracted at time t

- analogon to FRA but with continuous compounding
- link to zero bond prices:

$$R_t(S,T) :=$$

i.e.

$$\frac{P_t(S)}{P_t(T)} =$$

Model interest rates - but which?

Continuously compounded spot rate for $[t, T]$ at time t

- analogon to spot rate above but with continuous compounding
- link to zero bond prices:

$$R_t(T) :=$$

i.e.

$$\frac{1}{P_t(T)} =$$

Model interest rates - but which?

Instantaneous forward rate for time S contracted at time t

- What interest rate can I lock in today for a given investment held over the infinitesimal period $[S, S + dS]$ in the future $S > t$?
- analogon to FRA but for infinitesimal time period
- link to FRA and to zero bond prices:

$$F_t(S) = \lim_{T \downarrow S} R_t(S, T) :=$$

i.e.

$$P_t(T) =$$

- The mapping $[t, \infty) \in T \mapsto F_t(T)$ is called the forward curve at time t; it is sometimes beneficial to re-parameterize it in terms of the time-to-maturity $\tau = T - t$, i.e., in the form $[0, \infty) \in \tau \mapsto F_t(t + \tau)$.

Model interest rates - but which?

Instantaneous short rate/spot rate

- short term interest rate

$$r(t) = F_t(t) = \lim_{T \downarrow t} F_t(T) = - \frac{\partial}{\partial T}\Big|_{T=t} \log P_t(T)$$

- real-world proxy: overnight lending rate

Outline

Short rate models

Idea:

Model the dynamics of the "short rate" $(r_t)_{t \geq 0}$ on some filtered probability space $(\Omega, (\mathscr{F}_t), \mathbb{P}^*)$ and make the measure $\mathbb{P}^*$ an equivalent martingale measure for bond prices by putting

$$P_t(T) :=$$

Choose the parameters in the dynamics of $(r_t)_{t \geq 0}$ to calibrate your model to observed bond prices/term structure.

Examples

- Vasicek model

$$dr_t =$$

- Cox-Ingersoll-Ross (CIR) model

$$dr_t =$$

- Dothan model

$$dr_t =$$

! Constant coefficients $\rightsquigarrow$ rather inflexible, bad for calibration to market data for bond prices with many maturities

Examples

- Hull-White extension of Vasicek model

$$dr_t =$$

- Hull-White extension of CIR model

$$dr_t =$$

- Black-Derman-Toy model

$$dr_t =$$

- Ho-Lee model

$$dr_t =$$

Calibration

? *How to calibrate to market data, i.e., how to "invert the yield curve"?*

Example: Ho-Lee model

Recall

$$dr_t = \alpha(t)dt + \gamma dW_t^*$$

i.e.

$$r_t = r_s + \int_s^t \alpha(u)du + \gamma(W_t^* - W_s^*)$$

Calibrating the Ho-Lee model

Markovian structure gives PDE

? *How to calibrate when the distribution of $\int_t^T r_s ds$ is not easily computable?*

! Markov property of short rate models allows for the use of PDE method to compute bond prices.

Indeed:

$$P_t(T) = \mathbb{E}^* \left[\exp\left(- \int_t^T r_s ds \right) \middle| \mathscr{F}_t \right] = p(t, r_t, T)$$

where $p(.,.,T)$ should solve a linear PDE determined from the dynamics of (r_t) via Ito's formula.
Of course: Explicit solvability of the PDE will help tremendously, but may restrict flexibility.

Special case: affine models

Fortunately, there is a rich class of models which lead to nice PDEs and still retain a lot of flexibility:

Definition

If the function p in

$$P_t(T) = \mathbb{E}^* \left[\exp\left(-\int_t^T r_s ds \right) \middle| \mathscr{F}_t \right] = p(t, r_t, T)$$

is of the exponentially affine form

$$p(t, r, T) =$$

the associated model for (r_t) is called *affine* and we say that we have an *affine term structure model*.

?

- *How to tell whether p will be of this form?*
- *How to compute a and b?*
- *How does this really help with the calibration?*

Special case: affine models

! If short rate dynamics are of the form

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t^*$$

with

$$\mu(t, r) = \qquad\qquad\qquad \sigma(t, r) =$$

then the short rate model is affine.

Remark

Vasicek, CIR model along with their Hull-White extensions are affine, but *not* the Dothan and the Black-Derman-Toy models.

Special case: affine models

Illustration: Hull-White extension of Vasicek model

Determining a and b:

Illustration: Hull-White extension of Vasicek model

Calibration:

Outline

Forward rate models

Idea:

Model the dynamics of the yield curve $([0,\infty) \ni \tau \mapsto F_t(t+\tau))_{t\geq 0}$ as a function valued process starting in today's yield curve $\tau \mapsto F_0(\tau)$; obtain bond prices via

$$\frac{1}{P_t(T)} := \qquad\qquad\qquad .$$

Advantages:

- calibration to market data immediate
- flexible dynamics

Caveats:

- If we specify dynamics under real world measure $\mathbb{P}$, how to ensure no arbitrage?
- If we specify dynamics under pricing measure $\mathbb{P}^*$, how to ensure each discounted bond price process is a martingale?

Heath-Jarrow-Morton drift condition — for absence of arbitrage

Theorem

If forward rates follow dynamics of the form

$$dF_t(T) = \alpha_t(T)dt + \sigma_t(T)dW_t$$

for some d-dimensional Brownian motion W, the induced bond prices satisfy NFLVR (essentially) if and only if there is a d-dimensional process λ such that

$$\alpha_t(T) = \sigma_t(T) \cdot \int_t^T \sigma_t(S)dS - \sigma_t(T) \cdot \lambda_t$$

that does not depend on the maturity $T > 0$.

Heath-Jarrow-Morton drift condition — for absence of arbitrage

Heath-Jarrow-Morton drift condition — for martingale dynamics of discounted bond prices

Theorem

If forward rates follow dynamics of the form

$$dF_t(T) = \alpha_t^*(T)dt + \sigma_t(T)dW_t^*$$

for some d-dimensional $\mathbb{P}^*$-Brownian motion W^*, the induced bond prices $P_.(T) = (\exp(-\int_t^T F_t(S)dS))_{t\in[0,T]}$ will turn into $\mathbb{P}^*$-martingales when discounted essentially if and only if

$$\alpha_t^*(T) = \sigma_t(T) \cdot \int_t^T \sigma_t(S)dS.$$

Remark

Volatility structure $\sigma(.,.)$ under pricing measure fully determines drift $\alpha^*(.,.)$. Arbitrage free valuation thus only needs to specify volatility, just as in option pricing theory.

Heath-Jarrow-Morton drift condition
— for martingale dynamics of discounted bond prices

Heath-Jarrow-Morton drift condition — illustration

Take simplest possible volatility structure:

$$\sigma_t(T) \equiv \sigma \in (0, \infty)$$

Then:

Outline

Numéraires

So far we discounted asset prices by the evolution $\exp\left(\int_0^{\cdot} r_s ds\right)$ of a bank account earning interest at rate $(r_t)_{t\geq 0}$.
The idea of discounting is to make future asset prices comparable to today's prices as it measures the value of one asset in units of another one *at the same time*.
But of course this can be accomplished by any asset which has a strictly positive price process N. Such an asset is called a *numéraire*.

Example

Take $N_t := P_t(T)$, $t \in [0, T]$, the price process of a non-defaultable bond.
By no arbitrage: $N > 0$ until maturity.

? *How does option pricing change if we work with such a numéraire?*

Option pricing with change of numéraire

If $N > 0$ is the price process of a tradable asset, under any fixed pricing measure $\mathbb{P}^*$ it gives us a strictly positive martingale

$$\exp\left(-\int_0^t r_s ds\right) N_t, \quad t \in [0, T].$$

We can thus define a new probability measure $\mathbb{P}^N$ via

$$\left.\frac{d\mathbb{P}^N}{d\mathbb{P}^*}\right|_{\mathscr{F}_t} := \exp\left(-\int_0^t r_s ds\right) N_t / N_0, \quad t \in [0, T].$$

Clearly, if $H \geq 0$ is a time T-payoff to be priced, we can write

$$\pi_t(H) =$$

Special case: Forward measure

? *What does this look like for our example* $N = P_{\cdot}(T)$ *at* $t = T$*?*

We have

$$\left.\frac{d\mathbb{P}^T}{d\mathbb{P}^*}\right|_{\mathscr{F}_T} := \exp\left(-\int_0^T r_s ds\right) P_T(T)/P_0(T) =$$

and so

$$\pi_t(H) =$$

! Hence, when pricing time T-payoffs under $\mathbb{P}^T$ we do not have to discount anymore?!

Forward measure and forward price

Theorem

For a time T-payoff $H \geq 0$,

$$\mathbb{E}^T [H|\mathscr{F}_t] = \frac{\pi_t(H)}{P_t(T)}$$

is the forward price for H contracted at time $t \leq T$, i.e., it is the price $\mathrm{Fwd}_t^T(H)$ that is fixed at time t for delivery of H at time T against payment of $\mathrm{Fwd}_t^T(H)$ also at time T, without any intermediate payments at any time in $[t, T)$.

For obvious reasons:

Definition

The measure $\mathbb{P}^T$ is called *forward measure* for maturity T.

Clearly, forward prices of a time T-payoff H are martingales under the forward measure $\mathbb{P}^T$ with the same maturity—but not, in general, under forward measures $\mathbb{P}^{T'}$ with maturities $T' \neq T$.

Proof of forward price theorem

Forward formula for forward rates

Interestingly, we have

$$F_t(T) = \mathbb{E}^T \left[r_T | \mathscr{F}_t \right], \quad t \in [0, T].$$

So forward rates are best predictions of future short rates under the forward measure with the same maturity.

Indeed:

$$F_t(T) = -\partial_T \log P_t(T) = -\frac{\partial_T P_t(T)}{P_t(T)}$$

Futures

Since forward contracts generate cash flows only at maturity, the parties to the contract are exposed to **counterparty risk** as one of them might go bust in the meantime.

This is why forwards are traded only **"over the counter" (OTC)** where both sides in the transaction know each other. As a consequence, forwards cannot be traded freely, making them rather illiquid securities.

Futures alleviate the counterparty risk through an intermediary ("clearing house") that continually asks counterparties to post **margin payments** determined by the future price process.

Future prices

Definition

For a time T-payoff $H \geq 0$, the *future price* process $(\mathrm{Fut}_t^T(H))_{t\in[0,T]}$ is determined by the requirement that

$$\mathrm{Fut}_T^T(H) = H$$

and that a position of one future at time t exposes its holder to a payment of size $d\mathrm{Fut}_t^T(H)$ at this time and no further payments at all.

❓ *How can that be a definition?*

Future price theorem

Theorem

Under any pricing measure $\mathbb{P}^*$ *with numéraire* $\exp\left(\int_0^\cdot r_s ds\right)$ *where futures on* H *do not allow for arbitrage, we must have*

$$\mathrm{Fut}_t^T(H) = \mathbb{E}^*\left[H|\mathscr{F}_t\right], \quad t \in [0, T].$$

In particular, future prices are $\mathbb{P}^*$*-martingales.*

Corollary

For deterministic interest rates, future and forward prices coincide for any maturity.

Proof of the future price theorem

Part III

Financial optimization problems

Outline

Outline

Portfolio optimization in the Black-Scholes model

Consider a financial market with two investment opportunities:

- money market account bearing constant interest r
- stock with price process

$$S_t = s_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right),$$

$$X_t = e^{-rt} S_t = s_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right), \quad t \in [0, T],$$

for some Brownian motion W on $(\Omega, (\mathscr{F}_t), \mathbb{P})$.

? *How best to invest a given initial wealth $x > 0$?*

! Maximize the expected terminal wealth:

$$\mathbb{E}\left[V_T^{x,\theta}\right] \to \max_{\theta \text{ admissible}} \quad \text{where } V_T^{x,\theta} := x + \int_0^T \theta_s dX_s$$

Naive portfolio optimization problem ill-posed

But:

This optimization problem is ill-posed because

$$\sup_{\theta \text{ admissible}} \mathbb{E}\left[V_T^{x,\theta}\right] = +\infty$$

unless $\mathbb{P}$ is a martingale measure when the problem is pointless as the above supremum is x.

Remedy: Utility maximization

? *Why is this happening?*

!
- High expected return achieved by taking large risk: $|\beta|$ huge
- $V_T^{x,\theta} \to 0$ almost surely for $|\beta| \to \infty$: so pathwise typically abysmal performance compensated in expectation by getting incredibly rich in "a few" scenarios

$\rightsquigarrow$ We need to take into account the investor's **risk aversion**!

- Markowitz: Mean-Variance analysis
- Merton: Maximize expected utility which penalizes losses more than it rewards gains (cf. D. Bernoulli, St. Petersburg paradox)

Typical utility functions

- power utility

$$u(x) = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} & \text{for } x > 0 \\ -\infty & \text{for } x < 0 \end{cases} \text{ with } \alpha > 0,\ \alpha \neq 1$$

Remark: $\alpha = 1$ corresponds to log-utility: $u(x) = \log(x)$

- exponential utility

$$u(x) = -\exp(-\alpha x) \text{ with } \alpha > 0.$$

Merton problem:

Maximize $\mathbb{E}\left[u(V_T^{x,\theta})\right]$ over all admissible strategies θ!

Outline

The dynamic programming principle

Idea:

Do not focus directly on the optimal strategy but on the value it generates:

$$\mathbf{u}(T,x) := \sup_{\theta \text{ adm.}} \mathbb{E}[u(V_T^{x,\theta})]$$

This is called the *value function* (or indirect utility). Assess the performance of any strategy θ through its *value process*:

$$U_t^\theta := \mathbf{u}(T-t, V_t^{x,\theta}), \quad t \in [0,T].$$

This process describes what, at any time t, can still be accomplished if starting from the present state of affairs one controls the system optimally henceforth.

? *How to really assess performance in a random world?*

Martingale optimality principle

Intuition:

- Suboptimal strategies will tend to lead to worse values: U^{θ} is a supermartingale for all admissible θ.
- Optimal strategies will preserve value, at least on average: U^{θ^*} is a martingale for an optimal θ^*.

? *How to pin down the value function from these intuitive properties of value processes?*

Deriving the Hamilton-Jacobi-Bellman equation

Questions on the Hamilton-Jacobi-Bellman equation

?

- *Is this equation solvable? How?*
- *If we have found a solution, how to determine a candidate for an optimal strategy?*
- *How to prove that this candidate is actually optimal?*

Idea to prove optimality

Idea to find a candidate strategy

Idea to solve the HJB-equation in the power utility case

Verification theorem

Theorem

The value function of the Merton problem for power utilities is given by

$$\mathbf{u}(T,x) = \exp\left(\frac{1}{2}(1-\alpha)\frac{(\mu-r)^2}{\alpha\sigma^2}T\right)\frac{x^{1-\alpha}}{1-\alpha}, \quad x>0, T>0$$

and the optimal strategy is to always invest the same fraction

$$\pi^* = \frac{\mu-r}{\alpha\sigma^2}$$

of total wealth in stock.

Proof of the verification theorem

Outline

General utility maximization problem

Consider a **financial model** with two investment opportunities:

- money market account bearing predictable interest (r_s) with $\int_0^t |r_s| ds < \infty$, $t \geq 0$,
- stock with continuous price process (S_t) allowing NFLVR

both specified on a common filtered probability space $(\Omega, (\mathscr{F}_t), \mathbb{P})$.

Consider an **economic agent/investor** with

- initial capital $x > 0$,
- utility function u continuous, increasing and strictly concave on $[0, \infty)$ with $u(x) = -\infty$ for $x < 0$, $u(0) = 0$, and

$$u'(0+) = \infty, \quad u'(\infty) = 0 \quad (\text{"Inada-conditions"})$$

? *How to maximize the expected utility from terminal wealth*

$$\mathbb{E}\left[u(V_T^{x,\theta})\right] \to \max_{\theta \text{ admissible}}$$

when above expectation is set to $-\infty$ *if* $\mathbb{E}[u^-(V_T^{x,\theta})] = \infty$*?*

Mathematical challenges

Problems:

- Is there a solution to this optimization problem?
- If so, what can we say about it?
- How to overcome the lack of a Markovian structure which is key for dynamic programming and the approach via the Hamilton-Jacobi-Bellman PDE?

Remedy:

View the problem as a concave optimization problem and proceed via first order conditions which will not only be necessary but also sufficient for optimality!

The complete case: $|\mathscr{P}| = 1$

Assumption:

Suppose the market model is complete, i.e., all local martingales can be represented via integrals w.r.t. X, i.e., there is exactly one equivalent martingale measure $|\mathscr{P}| = 1$.

Lemma

Under the above completeness assumption, an $\mathscr{F}_T$-measurable random variable $H \geq 0$ is dominated by the discounted terminal wealth $V_T^{x,\theta}$ of some admissible strategy θ starting with initial capital $x > 0$ if and only if

$$\mathbb{E}^*[H] \leq x,$$

where $\mathbb{P}^*$ is the unique equivalent martingale measure in $\mathscr{P}$.

$\rightsquigarrow$ Instead of maximizing over predictable processes θ we can maximize over random variables $H \geq 0$!

The complete case: First order conditions

Standing assumption: $\mathbf{u}(x) := \sup_{\mathbb{E}^*[H] \leq x} \mathbb{E}[u(H)] < \infty$

Lemma

For $H^* \geq 0$ to be the terminal wealth of an optimal investment strategy with initial capital x it is necessary and sufficient that

$$\mathbb{E}^*[H^*] = x$$

and

$\mathbb{E}\left[u'(H^*)H\right] \leq \mathbb{E}\left[u'(H^*)H^*\right] < \infty$ *for any other* $H \geq 0$ *with* $\mathbb{E}^*[H] \leq x$.

$\rightsquigarrow$ Hence, an optimizer H^* for our **concave** optimization problem turns out to be also the optimizer for a **linear** optimization problem (with a single linear constraint to boot)!

Proof of first order conditions in the complete case

The complete case: Solution to first order conditions

Corollary

Under the Inada conditions, $H^* \geq 0$ *with*

$$\mathbb{E}^*\left[H^*\right] = x$$

will satisfy the first order conditions

$$\mathbb{E}\left[u'(H^*)H\right] \leq \mathbb{E}\left[u'(H^*)H^*\right] \text{ for any other } H \geq 0 \text{ with } \mathbb{E}^*\left[H\right] \leq x$$

if and only if there is $y > 0$ *such that*

$$u'(H^*) = y\frac{d\mathbb{P}^*}{d\mathbb{P}}, \text{ i.e., } H^* = (u')^{-1}\left(y\frac{d\mathbb{P}^*}{d\mathbb{P}}\right).$$

In particular, $H^y := (u')^{-1}\left(y\frac{d\mathbb{P}^*}{d\mathbb{P}}\right)$ *will be optimal for initial capital* $x > 0$ *if* $y > 0$ *is chosen such that* $\mathbb{E}^* H^y = x$*, which can actually be arranged uniquely.*

Proof of solution to first order conditions

Reconciling general findings with Merton's solution

? *Is the solution found via first order conditions the same as the one found via the solution of the Hamilton-Jacobi-Bellman equation in the Merton problem?*

Reconciling general findings with Merton's solution

The incomplete case

Let us now allow for $|\mathscr{P}| = \infty$ and try to proceed similarly as in the complete case.

? *Can we characterize the payoffs that can be dominated by an admissible strategy starting with a given initial capital?*

Super-replication theorem

Theorem

An $\mathscr{F}_T$*-measurable* $H \geq 0$ *satisfies* $H \leq x + \int_0^T \theta dX$ *for some admissible* θ *(i.e., it is super-replicable with initial capital* x*)* **if and only if**

$$\sup_{\mathbb{P}^* \in \mathscr{P}} \mathbb{E}^*[H] \leq x.$$

In fact, the super-replication price process for H *is of the form*

$$U_t := \operatorname*{ess\,sup}_{\mathbb{P}^* \in \mathscr{P}} \mathbb{E}^* \left[H | \mathscr{F}_t \right] = \sup_{\mathbb{P}^* \in \mathscr{P}} \mathbb{E}^*[H] + \int_0^t \theta dX - A_t, \quad t \in [0, T],$$

for some admissible θ *and some nondecreasing, rightcontinuous adapted* $A \geq 0$*.*

Remark

Note that the super-replication price process U is a supermartingale under **any** equivalent local martingale measure $\mathbb{P}^*$ for X, i.e., even a $\mathscr{P}$-supermartingale.

Optional decomposition

Theorem

A right-continuous process $U \geq 0$ is a $\mathscr{P}$-supermartingale if and only if

$$U_t = U_0 + \int_0^t \theta dX - A_t, \quad t \in [0, T],$$

for some admissible θ and some nondecreasing, rightcontinuous adapted $A \geq 0$.

Remark

This result can be viewed as a variant of the Doob-Meyer decomposition valid for any supermartingale. While it involves a right-continuous, increasing A that is only adapted (actually optional) rather than even predictable, it is more precise about the structure of the martingale part which is identified as a stochastic integral w.r.t. the local $\mathscr{P}$-martingale X. This is why the result is called *optional decomposition*. It is not unique though.

Proof of the optional decomposition theorem

Existence of an optimizer

Standing assumption: $\mathbf{u}(x) := \sup_{\mathbb{E}^*[H] \leq x \text{ for } \mathbb{P}^* \in \mathscr{P}} \mathbb{E}[u(H)] < \infty$

Theorem

If $\mathbf{u}$ *exhibits sublinear growth*

$$\limsup_{x \uparrow \infty} \frac{\mathbf{u}(x)}{x} = 0,$$

then for any initial capital $x > 0$ *there is an* $H^x \geq 0$*, uniquely determined up to a* $\mathbb{P}$*-null set, which solves the problem to*

Maximize $\mathbb{E}[u(H)]$ *over* $H \geq 0$ *such that* $\mathbb{E}^*[H] \leq x$ *for all* $\mathbb{P}^* \in \mathscr{P}$*.*

In particular, for any utility function which is bounded from above there is a unique solution to the corresponding utility maximization problem.

Proof

Characterizations of uniform integrability

Theorem

For any family of random variables $(X_i)_{i\in I}$ *on a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *the following assertions are equivalent:*

(i) $(X_i)_{i\in I}$ *is uniformly integrable.*

(ii) $\lim_{c\uparrow\infty} \sup_{i\in I} \mathbb{E}[|X_i| 1_{\{|X_i|\geq c\}}] = 0$

(iii) $(X_i)_{i\in I}$ *is bounded in* $L^1(\mathbb{P})$ *and for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\mathbb{E}[|X_i| 1_A] < \varepsilon$ *for any* $A \in \mathscr{F}$ *with* $\mathbb{P}[A] < \delta$.

(iv) $(\Phi(X_i))_{i\in I}$ *is bounded in* $L^1(\mathbb{P})$ *for some nondecreasing function* Φ *with* $\Phi(x)/x \nearrow \infty$ *as* $x \uparrow \infty$. *(de la Vallée-Poussin)*

(v) $(X_i)_{i\in I}$ *is weakly relatively compact in* $L^1(\mathbb{P})$. *(Dunford-Pettis)*

(vi) $(X_i)_{i\in I}$ *is bounded in* $L^1(\mathbb{P})$ *and for any decomposition* $\Omega = \cup_n A_n$ *of* Ω *into disjoint* $A_n \in \mathscr{F}$, $n = 1, 2, \dots$, *we have* $\lim_n \sup_{i\in I} \mathbb{E}[|X_i| 1_{A_n}] = 0$.

A version of Komlos's lemma due to Delbaen-Schachermayer

Lemma

For every sequence of random variables $X_n \geq 0$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, there are

$$\tilde{X}^n \in \mathrm{conv}(X^n, X^{n+1}, \dots), \quad n = 1, 2, \dots,$$

which almost surely converge to some random variable X with values in $[0, \infty]$.

Remark

This kind of property is sometimes referred to as *convex compactness*. It is remarkable that this is to be had for free despite the, in general, infinite dimensional setting where compactness is often hard to establish (or even hope for).

Proof

Construction of a solution in the incomplete case?

The first order condition in the incomplete case reads:

Lemma

For $H^* \geq 0$ to be the terminal wealth of an optimal investment strategy with initial capital x it is necessary and sufficient that

$$\sup_{\mathbb{P}^* \in \mathscr{P}} \mathbb{E}^* [H^*] = x$$

and

$$\mathbb{E}\left[u'(H^*)H\right] \leq \mathbb{E}\left[u'(H^*)H^*\right] < \infty \text{ for } H \geq 0 \text{ with } \sup_{\mathbb{P}^* \in \mathscr{P}} \mathbb{E}^* [H] \leq x.$$

$\rightsquigarrow$ Hence, an optimizer H^* for our **concave** optimization problem turns out to be also the optimizer for a **linear** optimization problem but with infinitely many linear constraints (in general).

$\rightsquigarrow$ No more general solution to write down, but . . .

The incomplete case: Convex duality

Observe that for any $H \geq 0$ with $H \leq x + \int_0^T \theta dX$ for some admissible θ and for any choice of martingale measure $\mathbb{P}^* \in \mathscr{P}$ and $y > 0$, we have

$$\mathbb{E}[u(H)] \leq$$

Duality gap

? *How to ensure "$=$" holds everywhere—no duality gap?*

Final remarks on the incomplete case

- theory with jumps considerably more involved
- existence for the dual problem via Komlos, but only in

$$\mathscr{D}(y) = \{D \geq 0 \ : D \leq Y_T \text{ for some } Y \geq 0 \text{ such that } YV^{1,\phi} \ \mathbb{P}\text{-supermart. for any adm. } \phi\}$$

  provided the dual problem is finite for any $y > 0$
- necessary & sufficient for existence and absence of duality gap without extra assumption on model: reasonable asymptotic elasticity

$$\limsup_{x \to \infty} \frac{u'(x)}{u(x)/x} < 1$$

  (rather than just ≤ 1 which always holds true !?!)
- for details see Kramkov&Schachermayer (1999, 2001)

Illustration: Utility maximization in Heston model

Heston model

- $dR_t := \frac{dX_t}{X_t} = \sqrt{\nu_t}dW_t$
- $d\nu_t = (\vartheta - \lambda\nu_t)dt + \gamma\sqrt{\nu_t}dB_t$
- (B, W) Brownian motions with correlation $[B, W]_t = \rho t$.

Power utility from terminal wealth

- $u(x) = x^{1-\alpha}/(1-\alpha)$ for some $\alpha > 0$, $u'(x) = x^{-\alpha}$, $x > 0$
- $V = V^{x,\pi}$ solves $V_0 = x$, $dV_t = \pi_t V_t dR_t$, i.e.,
  $V_t = x\mathscr{E}(\int \pi dR)_t$
- scaling argument: $\mathbf{u}(x) = x^{1-\alpha}\mathbf{u}(1) \rightsquigarrow$ w.o.l.g. $x = 1$

Optimal investment theorem for Heston model

Theorem

Define $a := -(1-(1-\alpha)/\alpha)\gamma^2\rho^2/2$, $b := \lambda - (1-\alpha)/\alpha\gamma\rho\mu$, $c := -\mu^2(1-\alpha)/(2\alpha)$ *and* $d := b^2 - 4ac = \lambda^2 - (2\lambda\gamma\rho\mu + \gamma^2\mu^2)(1-\alpha)/\alpha$ *and define* $c(t)$ *in case* $d > 0$ *as*

$$c(t) := -2c\frac{e^{\sqrt{d}(T-t)} - 1}{e^{\sqrt{d}(T-t)}(b+\sqrt{d}) - b + \sqrt{d}};$$

in case $d = 0$ *and either* $b > 0$ *or* $b < 0$, $T < -2/b$ *as*

$$c(t) := \frac{1}{a(T-t+2/b)} - \frac{b}{2a};$$

...

Optimal investment theorem for Heston model (ctd.)

Theorem

in case $d < 0$ *and either* $b > 0$, $T < 2(\pi - \arctan(\sqrt{-d}/b))/\sqrt{-d}$ *or* $b = 0$, $T < \pi/\sqrt{-d}$, *or* $b < 0$, $T < 2\arctan(\sqrt{-d}/-b))/\sqrt{-d}$ *as*

$$c(t) := -2c \frac{\sin(\sqrt{-d}(T-t)/2)}{\sqrt{-d}\cos(\sqrt{-d}(T-t)/2) + b\sin(\sqrt{-d}(T-t)/2)}.$$

Then in each of the above cases it is optimal to invest at time $t \in [0, T]$ *the fraction*

$$\hat{\pi}_t = \frac{\mu + \gamma\rho c(t)}{\alpha}$$

of one's total wealth in the stock; the resulting maximal expected utility as off time t *with* x *to spend and present vol* ν_t *is*

$$\mathbf{u}(x) = \mathbf{u}(t, x, \nu_t) = \exp\left(\int_t^T \vartheta c(s) ds + c(t)\nu_t\right) \frac{x^{1-\alpha}}{1-\alpha}.$$

Remarks

- When $\alpha \in (0,1)$, the problem may possibly degenerate in the sense that $\mathbf{u}(x) = \infty$; in fact, this happens precisely when the time horizon T is too large to fall in either of the cases distinguished above.
- For details see Kallsen and Muhle-Karbe, *Utility maximization in affine stochastic volatility models.* International Journal of Theoretical and Applied Finance, 13:459-477, 2010.

Outline

Merton's portfolio optimization problem

Some recent results in financial optimization