Basic Topology

- $B_r^d(x) := \{y \in X : d(x,y) < r\}$. $\bar{B}_r^d(x)$ may not $= \{y \in X : d(x,y) \le r\}$. e.g. $d(x,y) = \delta_{\{x=y\}}$ on $\mathbb{R}^1$.
$\Rightarrow \bar{B}_1(x) = \{x\}$. But $\{y \in X : d(x,y) \le 1\} = \mathbb{R}^1$.

Lem. (X, τ) topo space. Then A open $\Leftrightarrow \forall x \in A$. $\exists U$ nbd of x. s.t. $x \in U \subset A$.

Pf: $(\Rightarrow)$ Find $\tilde{U}$ open. $x \in \tilde{U}$. let $U = \tilde{U} \cap A$.

$(\Leftarrow)$ $A = \cup_{x \in A} U_x$ is open.

Def: (X, τ) topo space. $A \subset X$ is seq. closed if $\forall (x_n) \subset A \to x \in X$. $\Rightarrow x \in A$.

Rmk: In (X, d) metric space: A closed $\Leftrightarrow$ A seq closed. (By contradict)

Lem. Every metric space is Hausdorff

Pf: $\delta = d(x,y)$. $\forall x, y \in X$. Set $B(x, \frac{\delta}{3})$. $B(y, \delta/3)$ are two separating balls.

Lem. Limit is unique in Hausdorff

Pf: If $x_n \to x, \tilde{x}$. $\Rightarrow$ No open set of $x, \tilde{x}$ separate them. ($\exists x_n \in U_x \cap U_{\tilde{x}}$)

Lem. $\{x\}$ is closed in Hausdorff space X.

Cor. Cpt sets are closed in X.

e.g. i) Cocountable topo: $X = \mathbb{R}^1$. $\tau = \{A \subset X :$ $A = \phi$ or A^c is countable $\}$.

a) closed $\not\Leftrightarrow$ seq closed.

closed set: $A = X$ or countable set.

seq closed set: $\forall A \subset X$ is seq closed

Note that for $x_n \subset A \to x$. let $U = \tilde{U} / (\{x_n\} / \{x\})$. $\tilde{U}$ is nbd of x.

So: $\exists N$. $(x_n)_N^\infty \subset U$. $\Rightarrow x_n = x$. $\forall n > N$.

$\Rightarrow x \in A$. for $\forall A \subset X$.

b) (X, τ) isn't metrizable.

otherwise, it's Hausdorff. But for $x, y \in X$. if $\exists A, B$ open $x \in A \cap B \ni y = \phi$.

$\Rightarrow A^c \cup B^c = X$. contradict!

ii) Cofinite topo: $X = \mathbb{R}^1$. $\tau = \{A \subset X \mid A = \phi$

or A^c is finite set $\}$.

a) (X, τ) isn't Hausdorff. (as i) b))

b) $\forall$ set $\subset (X, \tau)$ is cpt and seq. cpt.

$N \subset \bigcup U_i$. Assume $\exists U_i$. s.t. $N \cap U_i^c \neq \emptyset$.

so $N \cap U_i^c = \{x_1, \cdots x_m\}$. $\exists U_{i_k} \ni x_k$. $\forall k$.

$\Rightarrow N \subset U_i \cup (\bigcup_1^m U_{i_k})$.

And any infinite distinctive seq will enter open sets eventually.

Lem. i) (E, d) is separable $\Rightarrow \forall F \subset E$. $(F, d_{F \times F})$ is separable.

ii) $A \subset (E, d)$ is cpt. $\Leftrightarrow (A, d_{A \times A})$ is cpt.

Pf: i) let $y_n^k \in F \cap B_{\frac{1}{k}}(x_n)$ if $\neq \emptyset$. for (x_n) is countable dense in E.

$\Rightarrow (y_n^k)_{k,n}$ is dense in $(F, d_{F \times F})$.

ii) Note V is open in $(A, d_{A \times A}) \Leftrightarrow \exists U$ open in E and $V = A \cap U$.

Lem. $(E, \|\cdot\|)$ is separa. $\Leftrightarrow S_E = \{\|x\| = 1\}$ is.

Pf: $(\Rightarrow)$ is by Lem i). $(\Leftarrow)$ $(x_n) \underset{\text{dense}}{\subset} S$.

We assert $(\lambda x_n)_{\lambda \in \mathbb{Q}, n} \underset{\text{dense}}{\subset} E$.

$\forall x \in E. \Rightarrow \frac{x}{\|x\|} \in S.$ $\exists \|x_n - \frac{x}{\|x\|}\| < \frac{\varepsilon}{\|x\|}$

and $\exists \lambda \in \mathbb{Q}. \ |\lambda - \|x\|| < \varepsilon$.

Lem. E, n.v.s. Then: E^* sep. $\Rightarrow E$ sep.

Pf: (l_n) dense in S_{E^*}. Choose (x_n) s.t.

$\|x_n\| = 1$. $|l_n(x_n)| \geq \frac{1}{2}$.

Claim: $\mathrm{span}^{\mathbb{Q}}\{x_k\}_{k \geq 1} := F$ dense in E.

If F isn't dense. Then: $\exists l \in E^*$.

s.t. $l|_F = 0$. $\|l\| = 1$. So:

$\frac{1}{2} \leq |l_n(x_n)| = |l_n(x_n) - l(x_n)| \leq \|l_n - l\|$

$\Rightarrow$ contradiction.

Def: $A \subset (X, d)$ metric space is totally bdd if $\forall \varepsilon > 0$. $\exists$ finite ε-net for A. (i.e. $A \subset \bigcup_i B_j(\varepsilon)$)

Thm. For (X, d) metric space.

i) X is complete and totally bdd.

ii) X is cpt. iii) X is seq. cpt.

Then: i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii).

Pf: i) $\Rightarrow$ ii) By contradiction. $\exists \{U_i\}$ cover of X doesn't have finite subcover.

Inductively define $\{x_n\}$ s.t.

a) $B_{2^{-n}}(x_n)$ isn't covered by finite u_i

b) $B_{2^{-n}}(x_n) \cap B_{2^{-(n+1)}}(x_{n+1}) = \emptyset$.

Note X is totally bdd. So $\exists |M_1| < \infty$. s.t.

$X = \bigcup_{M_1} B_{\frac{1}{2}}(y)$. $\Rightarrow \exists x_1 \in M_1$ s.t. $B_{\frac{1}{2}}(x_1)$ won't be covered by finite u_i.

Otherwise $\{u_j\}$ admits finite covering.

For $X = \bigcup_{M_{n+1}} B_{2^{-(n+1)}}(y)$. If any $y \in M_{n+1}$ s.t.

$B_{2^{-(n+1)}}(y) \cap B_{2^{-n}}(x_n) \neq \emptyset$ will also satisfy

$B_{2^{-(n+1)}}(y)$ is covered by finite u_i.

$\Rightarrow B_{2^{-n}}(x_n)$ is covered by finite u_i. which is contradicted with induct hypothesis.

Let $z_n \in B_{2^{-n}}(x_n) \cap B_{2^{-(n+1)}}(x_{n+1})$. $\forall n$

$$\Rightarrow d(x_k, x_m) \le \sum_k^m d(x_{j+1}, x_j)$$

$$\le \sum_k^m (d(x_{j+1}, z_j) + d(x_j, z_j))$$

$$\lesssim 2^{-k}. \text{ for } k < m.$$

So: $\{x_n\}$ is Cauchy $\to x^* \in X$.

Choose $u \in \{u_i\}$. s.t. $x^* \in u$. $\Rightarrow \exists n$ large

s.t. $B_{2^{-n}}(x_n) \subset u$. Contradiction.

ii) $\Rightarrow$ iii) By contradiction: if $(x_n) \subset X$ doesn't have

convergent subseq. then:

$\forall x \in X$. $\exists \varepsilon_x, n_x$. s.t. $x_n \notin B_{\varepsilon_x}(x)$. $\forall n \geq n_x$

But $X \subset \bigcup_i^{\sim} B_{\varepsilon_{x_i}}(x)$ by cpt.

So: $x_n \notin X \subset \bigcup_i^{\sim} B_{\varepsilon_{x_i}}(x)$. $\forall n \geq \max_i n_{x_i}$.

iii) $\Rightarrow$ i) Lem. Cauchy seq. having a convergent

subseq in (X, d) will converge

By Lem.: X is complete.

If X can't be covered by finite $B_\varepsilon(x)$

$\Rightarrow \exists (x_n)$. s.t. $x_n \notin B_\varepsilon(x_j)$. $j \leq n-1$.

So: (x_n) won't contain convergent subseq

Cor. cpt $A \subset (X, d)$ is bdd. closed.

complete and separa

Pf: bdd and separa. is from A

is totally bdd

$\Rightarrow \bigcup_{i=1}^{m_n} \{x_i^n\}_1^{m_n}$ is dense. ($\{x_i^n\}_1^{m_n}$ is $\frac{1}{n}$-net)

Rmk: Conversely, bdd. closed $\not\Rightarrow$ cpt.

e.g., $X = \{1/n\}_{\mathbb{N}}$. $d(x,y) = \delta_{\{x=y\}}$. (X,d) is bdd. closed.

But $X \subset \bigcup_n B_{\frac{1}{2}}(1/n)$ has no finite subcovering.

Lem. For X metric space. $A \subseteq X$. Then:

i) X is totally bdd $\Rightarrow$ A is totally bdd

ii) A is totally bdd $\Rightarrow$ $\bar{A}$ is totally bdd

Cor. For X complete metric space:

pre-cpt $\Leftrightarrow$ totally bdd.

Pf: i) $\{x_i\}_1^m$ is $\frac{\varepsilon}{2}$-net for X. $\Rightarrow$

$\{y_j \mid y_j \in A \cap B_{\frac{\varepsilon}{2}}(x_j)\}_1^m$ is ε-net for A

ii) $\{x_i\}_1^m$ is $\frac{\varepsilon}{2}$-net for A $\Rightarrow$

$\{x_i\}_1^m$ is ε-net for $\bar{A}$

Thm (Ascoli-Arzelà)

(X,d) is cpt. $F \subset (C(X), \|\cdot\|_\infty)$. Then:

i) F equiconti. & pointwise bdd $\Rightarrow$ pre-cpt

ii) F is precpt. $\Rightarrow$ uniform equiconti. and uniform bdd.

Pf: i) $\forall \varepsilon$, $\exists \delta_x$ st. $|f(y)-f(x)| < \varepsilon/4$. for $\forall f \in F$ and $\forall y \in B_{\delta x}(x)$.

And $X = \bigcup_1^m B_{\delta_{x_i}}(x_i)$ from X is cpt.

Set $K = \bigcup_1^m F(x_i) \subset \bigcup_1^n B_{\varepsilon/4}(\lambda_i)$.

Since F is pointwise bdd.

let $\Phi = \{ \varphi : \{1, \cdots m\} \mapsto \{1, \cdots n\} \}$ finite

Claim $F = \bigcup_{\varphi \in \Phi} F_\varphi$. $F_\varphi = \{ f \in F \mid |f(x_i) - \lambda_{\varphi(i)}| \le \frac{\varepsilon}{4}. \forall 1 \le i \le m \}$

and prove $d(f, g) < \varepsilon$. $\forall f, g \in F_\varphi$.

So: choose $f_\varphi \in F_\varphi$. $\Rightarrow$ $\{f_\varphi\}$ is ε-net.

ii) F is totally bdd $\Rightarrow$ uniformly bdd.

And if $\exists \varepsilon_0 > 0$. st. $\forall n$, $\exists$ $d(x_n, y_n) < \frac{1}{n}$,

$\exists f_n \in F$. st. $|f_n(x_n) - f_n(y_n)| > \varepsilon_0$.

But (f_n) has convergent subseq. $\xrightarrow{\text{u.l.a.}} f$.

$LHS \le |f_n(y_n) - f(y_n)| + |f(y_n) - f(x_n)| + |f(x_n) - f_n(x_n)|$

$\leq 2\|f_n - f\|_\infty + |f(y_n) - f(x_n)| \to 0.$

$\Rightarrow$ it's contradiction.

Cr. $X = [0,1] \Rightarrow \dim C(X) = +\infty$.

Pf: $K = \{f \in C(X) \mid \|f\|_\infty \leq 1\}$ isn't cpt

: $g_n(x) = x^n \to g(x) = \delta_1 \notin C(X)$.

So it's also not equiconti.

Thm. (Baire)

(X, d) is complete, (U_n) is seq of open dense $\subset X$. $\Rightarrow \bigcap_n U_n$ is open, dense.

Pf: $\forall x \in X$. $\forall r > 0$ fix. We prove:

$B_r(x) \cap (\bigcap_n U_n) \neq \emptyset$.

1) By density of U_n. $\forall n$. let $r_n = 2^{-n} r$

$\Rightarrow \exists (x_n)$. s.t. $D_{r_n}(x_n) \subset\subset U_{n-1} \cap B_{r_{n-1}}(x_{n-1})$

where $x_0 = x$. $r_0 = r$. iteratively def.

2) So: (x_n) is Cauchy $\Rightarrow x_n \to x^*$.

Then $x^* \in \bigcap \overline{B}_{r_n}(x_n) \subset B_r(x) \cap (\bigcap U_n)$

Thm. $X = (C[0,1], \|\cdot\|_\infty)$. Nowhere differentiable functions are dense in X.

pf: 1) $A_k := \{ g \in X \mid \forall t \in [0,1].\ \sup_{0<|h|<k^{-1}} |g(t+h) - g(t)| / |h| > k \}$.

$\forall f \in \bigcap_k A_k \Rightarrow f$ is nowhere diff.

So we want to apply Baire Thm on (A_k).

2) prove: A_k^c is closed

For $(f_n) \subset A_k^c \to f$. Note $\forall n$.

$\exists t_n \in [0,1]: \sup_{0<|h|<k^{-1}} |f_n(t_n+h) - f_n(h)|/h \leq k$.

Note $\exists t_{n_k} \to \tilde{t} \in [0,1]$. By triangle inequi. $\Rightarrow |f(\tilde{t}+h) - f(\tilde{t})|/h \leq k$. $\forall h < k^{-1}$.

So: $f \in A_k^c \Rightarrow A_k^c$ is closed.

3) prove: A_k is dense in X.

$\forall f \in X$. Note f is unif. conti. on $[0,1]$

We do partition $\pi = \{ t_i = i/N.\ i = 0, \dots N \}$

let g is linear intepolation for f on π. s.t. $g(t_i) = f(t_i) + (-1)^i \varepsilon$.

Choose $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. St.

$N^{-1} < (N\varepsilon)^{-1} < \delta < k^{-1}$. $\Rightarrow \|f-g\|_\infty < 4\varepsilon$

And $|g(t_i) - g(t_{i+1})| / N^{-1} \geq \varepsilon N > k$.

<u>Def</u>: d_1, d_2 metric on X. We say $d_1 \sim d_2$ if open sets of (X, d_1), (X, d_2) coincide.

<u>Lem</u>: $d_1 \sim d_2 \Leftrightarrow$ converging seq are same in (X, d_1), (X, d_2).

<u>Pf</u>: $(x_n) \to x \in (X, d_1)$. $\Leftrightarrow \forall U$ open nbd of x. $\exists N$. $(x_n)_N^\infty \subset U$. $\Leftrightarrow$ $(x_n) \to x \in (X, d_2)$.

<u>e.g.</u> $X = (0,1]$. $d_1(x,y) = |x-y|$. $d_2(x,y) = |x^{-1} - y^{-1}| \Rightarrow$ $d_1 \sim d_2$. But (X, d_1) isn't complete since $(\frac{1}{n})$ Cauchy in (X, d_1). but $\to 0 \notin X$. while (X, d_2) is complete $\cong ([1,\infty), d_1)$

<u>Rmk</u>: i) Completeness isn't topo. property.

But Baire property is:

If $(G_n) \subset (X, d_1)$. dense. open $\Rightarrow$ (G_n) also dense. open in (X, d_2).

While we can apply Baire Thm on (X, λ_2). $\Rightarrow \bigcap G_n$ dense, open in (X, λ_2) so also in (X, λ_1).

ii) If λ_1, λ_2 is induced by norms $\|\cdot\|_1$, $\|\cdot\|_2$. Then: $\lambda_1 \sim \lambda_2 \Leftrightarrow \|\cdot\|_1 \sim \|\cdot\|_2$

So: $\lambda_1 \sim \lambda_2 \Leftrightarrow (X, \lambda_1), (X, \lambda_2)$ are or aren't complete simultaneously.

<u>e.g.</u> $(C[0,1], \|\cdot\|_{L^1[0,1]})$ isn't complete:

Note $f_n(x) = nx I_{\{0 \le x \le \frac{1}{n}\}} + I_{\{n^{-1} \le x \le 1\}}$ is L^1-Cauchy. But $f_n \to f = I_{(0,1]} \notin C[0,1]$.

$(C[0,1], \|\cdot\|_{L^1[0,1]})$ isn't completely metrizable:

Note $A_n = \{f \in C[0,1] \mid \|f\|_\infty > n\}$ is open & dense ($\forall f \in C[0,1]$. let $g = f + M \cdot I_{[0, \frac{1}{M}]}$)

But $\bigcap A_n = \emptyset$. $\Rightarrow$ Baire Thm fails.

<u>Rmk</u>: A_n isn't dense in $(C[0,1], \|\cdot\|_\infty)$ which's complete! (e.g. $f \equiv 0$).