Normed Spaces Examples

(1) Sequence spaces:

Def: $c_{00} := \{(x_n) \mid x_n \neq 0$ only for finite $n\}$.

$c_0 := \{(x_n) \mid \lim_{n\to\infty} |x_n| = 0\}$

$c := \{(x_n) \mid \lim x_n$ exists$\}$.

$\ell_\infty := \{(x_n) \mid (x_n)$ is bdd$\}$.

$\ell_p := \{(x_n) \mid (\sum |x_n|^p)^{\frac{1}{p}} = \|x\|_p < \infty\}$. $1 \leq p < \infty$.

Rmk: $c_{00} \subset c_0 \subset c \subset \ell_\infty$; $\ell_1 \subset \ell_2 \cdots \subset \ell_\infty$.

Thm. ℓ_∞ isn't separable.

Pf: $M = \{(x_n) \mid x_n \in \{0,1\}\}$ isn't countable.

$\forall x \neq y \in M$, $\|x - y\|_\infty = 1$

If $\exists S = (x^n)$ dense in ℓ_∞. Then:

$\|x^{n_1} - y\|_\infty < \frac{1}{2}$. $\|x^{n_2} - x\|_\infty < \frac{1}{2}$. $\forall x, y \in M$.

$\Rightarrow \|x^{n_1} - x^{n_2}\| > 0$. So: $x^{n_1} \neq x^{n_2}$.

$x \mapsto x^{n_1}$ is injective. $\Rightarrow \# M \leq \# S$.

Thm. ℓ_p is separable. $\forall 1 \leq p < \infty$.

Pf: let $M = \bigcup_{k=1}^{\infty} \mathbb{Q}^k \times \{0\} \times \cdots \{0\} \times \cdots$

Note $\forall x \in \ell_p$. $\exists N$. $\sum_{N+1}^{\infty} |x(k)|^p \le \varepsilon^p/2$.

Also $\exists (q_k)$. $\sum_1^{N} |x(k) - q_k|^p \le \varepsilon^p/2$

Thm. ℓ_p and c_0 is complete. $\forall 1 \le p \le \infty$.

Pf: $1 \le p < \infty$: (x_n) is ℓ_p-Cauchy.

$\Rightarrow |x_n(k) - x_m(k)| \le \|x_n - x_m\|_{\ell_p}$

So: $(x_n(k))_n$ is Cauchy in $\mathbb{K}$.

$\sum_{k=1}^{t} |x_m(k) - x_n(k)|^p \le \|x_m - x_n\|^p \le \varepsilon^p$. for

$\forall m, n > N$. $\Rightarrow$ let $m \to \infty$. then $t \to \infty$.

For $p = \infty$. it's analogous.

Thm. $\dim \ell_p = +\infty$.

Pf: $(e_n) \subset \ell_p$. $e_n = (0, \cdots \overset{n^{th}}{1}, 0 \cdots 0 \cdots)$. has

no ℓ_p-convergent subseq. Otherwise:

$e_{n_k}(r) \to 0$. $\forall r$. So: $e_{n_k} \to 0$.

But $\|e_{n_k} - 0\|_{\ell_p} = 1$. So $\bar{B}_1$ isn't cpt.

Thm. i) $(c, \|\cdot\|_\infty) \simeq (c_0 \oplus \mathbb{K}, \|\cdot\|_\infty + |\cdot|) \simeq (c_0, \|\cdot\|_\infty)$

ii) $c_{00}^* \cong (c_0)^* \cong \ell_1$. iii) $c^* \cong \ell_1$.

Rmk: ("$\simeq$" is isomor.; "$\cong$" is "$\underset{\text{isometric}}{\simeq}$")

$c \simeq c_0$. $c^* \cong c_0^*$ from above.

But $c \not\cong c_0$.

Pf: i) $T: (x_n) \in c \mapsto ((x_n - \lim x_n)_n, \lim x_n) \in c_0 \oplus \mathbb{K}$.

is isomorphism.

$S: ((x_n), \lambda) \in c_0 \oplus \mathbb{K} \mapsto (\lambda, x_1, x_2 \cdots) \in c_0$.

is isomorphism.

ii) **Lem.** $u \underset{\text{dense}}{\subset} E \Rightarrow u^* \cong E^*$.

Pf: By conti. extension. $\forall l \in u^*$.

$\exists! \bar{l} \in E^*$. $\bar{l}|_u = l$. $\|\bar{l}\| = \|l\|$.

let $T: l \in u^* \mapsto \bar{l} \in E^*$.

Set $T: y \in l_1 \mapsto f_y \in c_0^*$. where

$f_y(x) = \sum_1^\infty y_n x_n$. $\forall x \in c_0$. is well-def.

And $\|f_y\| = \|y\|_{l_1}$ ("$\leq$" by Hölder;

"$\geq$" by let $(x_n) = (\bar{y}_n/|y_n| \cdot I_{y_n \neq 0, n \leq N})$

$\Rightarrow T$ is isometry. For surjective:

$\forall f \in c_0^*$. let $(y_n) = (f(e_n))_n$.

$\sum_1^m |y_n| = \sum_1^m \frac{\bar{y}_n}{|y_n|} \cdot y_n = f(\cdots) \leq \|f\|$.

So $(\eta_n) \in \ell_1$. With $(x_1, \cdots x_m, 0 \cdots 0)$

$\xrightarrow{\|\cdot\|_\infty} x = (x_n)$ (Here we use c_0!)

$\Rightarrow f(x) = \sum x_n f(e_n) = f_\eta(x)$ by conti.

iii) $T: \eta \in \ell_1 \mapsto f_\eta \in c^*$. where $f_\eta(x) :=$

$\eta_1 \lim x_n + \sum x_n \eta_{n+1} \in c^*$ is well-def

And $\|f_\eta(x)\| \le \|x\|_\infty \|\eta\|_1$

Set $\tilde{x_n} = \bar{\eta}_{n+1} / |\eta_{n+1}| \, I_{\{n \le N, \eta_{n+1} \ne 0\}} + \frac{\bar{\eta}_1}{\eta_1} I_{\{n > N, \eta_1 \ne 0\}}$

$\Rightarrow |f_\eta(\tilde{x_n})| \ge \sum_1^{\tilde{}} |\eta_n| - \sum_{N+1}^{\infty} |\eta_n| \xrightarrow{N \to \infty} \|\eta\|_{\ell_1}$

So: $\|f_\eta\| = \|\eta\|_{\ell_1} \Rightarrow T$ is isometry.

For surjective: $\forall g \in c^*$.

Note $g(x) = \sum_{n \ge 1} x_n g_{n+1}$. $\forall x \in c_0$. by ii)

let $\eta_1 = g(e) - \sum g_{n+1}$. $e = (1, 1, \cdots 1, \cdots)$

So: $g(x) = g(x - \lim x_n \cdot e) + \lim x_n \, g(e)$

$= \sum (x_n - \lim x_n) g_{n+1} + \lim x_n (\eta_1 + \sum g_{n+1})$

$= \sum x_n g_{n+1} + \eta_1 \lim x_n = f_\eta(x)$

where $\eta = (\eta_n)_{n \ge 1} \in \ell_1$.

Thm. $(\ell_p)^* \cong \ell_q$. $\forall 1 < p < \infty$. $1/p + 1/q = 1$.

Pf: $T: y \in \ell_q \mapsto f_y \in \ell_p^*$ defined by

$f_y(x) = \sum x_n y_n \Rightarrow \|f_y\| \le \|y\|_{\ell_q} \Rightarrow$ well-def

i) Prove: $\|f_y\| = \|y\|_q$.

$p=1$: $\exists n.\ |y_n| > \|y\|_\infty - \varepsilon$. So:

$|f_y(e^n)| > \|y\|_\infty - \varepsilon \to \|y\|_\infty$.

$p>1$: Set $(x_n) := \left(\frac{|y_n|^q}{y_n} I_{\{y_n \ne 0\}}\right) \in \ell_p$

And $|f_y(x)| = \|y\|_q^q$.

2) $\forall \ell \in \ell_p^*$. Set $y_n = \ell(e^n)$.

$\Rightarrow (y_n) \in \ell_q$ and $\ell(x) = f_y(x)$ on

$\mathrm{span}(e^n) \Rightarrow \ell = f_y$.

Rmk: We use map $T: y \in \ell_q \mapsto f_y \in \ell_p^*$

defined by $f_y(x) = \sum x_n y_n$ in proof

When $q=1$: T is linear isometry but

not surjective: e.g.

$\ell: c \to \mathbb{R}^1.\ x \mapsto \lim x_n$. By Hahn-Banach

$\exists L \in \ell_\infty^*.\ L|_c = \ell.\ |L(x)| \le \|x\|_\infty$

If $\exists y \in \ell_1$. s.t. $L = f_y$. Then:

$L(e^n) = 0 = f_y(e^n) = y_n \Rightarrow y = 0.\ L = 0.$

Rmk: $l_1 \neq l_\infty^*$ since l_∞ isn't separa. but l_1 is. (E^* sep. $\Rightarrow$ E sep.)

Thm l_p is reflexive. $\forall 1<p<\infty$.

Pf: $\forall x'' \in l_p^{**}$. Set $T: l_p \cong l_q^*$ and $S: l_q \cong l_p^*$ are Riesz isomor. above.

let $x = T^{-1} S^* x'' \Rightarrow x''(\eta) = \eta(x)$. $\eta \in l_p^*$

Lem. $(e^n) \rightharpoonup 0$ in l^p. $1<p<\infty$

Rmk: i) $e^n \not\to 0$. by $\|e^n\|_p = 1$.

ii) $e^n \not\rightharpoonup 0$ in l_1. since let $(\eta_n) = (I_{\{2|n\}}) \in l_\infty \Rightarrow e^n(\eta)$ diverges.

Def: $L: l_\infty \xrightarrow{\text{linear}} \mathbb{R}'$ is Banach limit if

a) $L(Tx) = L(x)$. $\forall x \in l_\infty$, $T: (x_1, x_2 \cdots) \mapsto (x_2, x_3 \cdots)$

b) $L(x) \geq 0$ if $x = (x_n)$. $x_n \geq 0$. $\forall n$.

c) $L(1,1,\cdots) = 1$.

prop. i) There exists Banach limit

ii) Banach limit $L: l_\infty \to \mathbb{R}'$ satisfies:

a) $L \in l_\infty^*$. $\|L\| = 1$

b) $\underline{\lim} x_n \le L(x) \le \overline{\lim} x_n$. $\forall x \in l_\infty$.

c) $L(1,0,1,\cdots) = \frac{1}{2}$.

<u>Rmk</u>: Banach limit L isn't unique.

<u>Pf</u>: i) Set $l : x \in c \mapsto \lim x_n \in \mathbb{R}^1$.

$p : x \in l_\infty \mapsto \overline{\lim} \frac{1}{n}\sum_1^n x_k \in \mathbb{R}^1$

$|p(x)| \le \|x\|_\infty < \infty$. $\Rightarrow p \in l_\infty^*$.

And $p|_c = l$. by Stolz Thm.

$\Rightarrow p$ is sublinear dominate l on c

By Hahn-Banach. $\exists L \in l_\infty^*$. $L|_c = l$.

$L(x) \le p(x)$. $\Rightarrow -L(x) = L(-x) \le p(-x) \le 0$ for

$x = (x_n)$. $x_n \ge 0$. $\forall n$.

$L(x-Tx), L(Tx-x) \le p(x-Tx) = 0 = p(Tx-x)$

So: $L(x) = L(Tx)$.

And $L(1,\cdots 1 \cdots) = l(1,\cdots 1\cdots) = 1$

ii) a) $L(\|x\|_\infty, \|x\|_\infty, \cdots) = \|x\|_\infty L(1,\cdots 1, \cdots) = \|x\|_\infty$.

And $L(\|x\|_\infty, \|x\|_\infty \cdots) - L(x) = L(\|x\|_\infty - x_n) \ge 0$

So: $L(x) \le \|x\|_\infty$. i.e. $\|L\| \le 1 \Rightarrow \|L\| = 1$

b) $\exists N$. $\forall n > N$. $\underline{\lim} x_k - \varepsilon < x_n < \overline{\lim} x_k + \varepsilon$.

By shift-invar. : $\overline{\lim} x_k + \varepsilon - L(x_1, x_2 \cdots)$

$= \boxed{} - L(x_N, x_{N+1} \cdots) \geq 0$.

Similarly : $\underline{\lim} x_k - \varepsilon < L(x) < \overline{\lim} x_k + \varepsilon$.

c) $2L(1,0,1,0 \cdots) = L(1,0,1,0 \dots) + L(0,1,0,1 \cdots)$

$= L(1,1,\cdots) = 1$.

(2) L^p Space:

Young inequi. $\Rightarrow$ Hölder inequi.

$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ $\qquad a = f/\|f\|_p$. $b = g/\|g\|_q$

Thm. L^p is complete.

Pf: i) $\exists (f_{n_k}) \subset L^p$. $\|f_{n_k} - f_{n_{k+1}}\|_p \leq 2^{-k}$.

2) So : $\sum_1^m (f_{n_{k+1}} - f_{n_k}) \xrightarrow{a.s.} f \in L^p$.

$$\int |f_k - f|^p = \int \underline{\lim}_m |f_k - f_{n_m}|^p$$

$$\overset{\text{Fatou's}}{\leq} \underline{\lim}_m \int |f_k - f_{n_m}|^p \leq \varepsilon^p$$

Thm. (Riesz Repre.)

$(L^p(\mu))^* \cong L^q(\mu)$. $1 \leq p < \infty$. $1/p + 1/q = 1$.

Pf: $T : g \in L^q \mapsto (Tg)(f) := \int gf \in (L^p)^*$.

1°) Isometry : $|Tg(f)| \leq \|f\|_p \|g\|_q$.

And let $f = \frac{\bar{g}}{|g|} \cdot \left(\frac{|g|}{\|g\|_2}\right)^{2/p}$

ii) Surjective: $\forall \ell \in (L^p)^*$. $V(A) = \ell(I_A)$

is abs. conti. w.r.t $\mathcal{L}$

$\Rightarrow$ RN Thm: $\exists g \in L^1$. $\ell(I_A) = \int_A g \, dx$

$\Rightarrow$ extend on L^p: $\ell(f) = \int f g \, dx$

Next, prove: $g \in L^q$.

Set $A = [|g| > n]$. $n > \|\ell\|$.

$p = 1$: $|A| \|\ell\| < \int_A |g| \, dx = \ell(I_A \frac{|g|}{g})$

$\leq |A| \|\ell\| \Rightarrow |A| = 0$.

$p > 1$: $\int_{A^c} |g|^q \leq \|\ell\| (\int_{A^c} |g|^q)^{1/p}$.

By MCT. let $n \to \infty$. $\Rightarrow \|g\|_q \leq \|\ell\|$

Lem. For $1 \leq p < r < q < \infty$. $\theta \in (0,1)$. $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$

Then: $L^r(\Omega) \supseteq L^p(\Omega) \cap L^q(\Omega)$. Besides

$\|f\|_r \leq \|f\|_p^{\theta} \|f\|_q^{1-\theta}$.

Rmk: $L^p(\Omega) \nsubseteq L^q(\Omega)$ nor $L^q(\Omega) \nsubseteq L^p(\Omega)$

in general. e.g. $\Omega = \mathbb{R}^1$.

$x^{-p^{-1}} I_{[1,\infty)} \in L^q(\mathbb{R}^1)$. $\notin L^p(\mathbb{R}^1)$. $\forall q > p$.

$x^{-q^{-1}} I_{(0,1]} \in L^p(\mathbb{R}^1)$. $\notin L^q(\mathbb{R}^1)$. $\forall q > p$.

Lem. $\dim L^p(0,1) = +\infty$. $\forall 1 \le p \le \infty$.

Pf: $(f_n) := (n^{p^{-1}} I_{(0,n^{-1})}) \subset \bar{B}$, won't have L^p-convergent subseq.

(Note $f_n \to 0$ a.s. $\|f_n\|_{L^p} = 1$.)

Thm. $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is complete.

Pf: $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty \to 0$.

$\Rightarrow \exists f(x)$. $f_n \to f$ pointwise.

$|f(x) - f_n(x)| \leftarrow |f_m(x) - f_n(x)| \le \varepsilon$.

$\Rightarrow \|f - f_n\|_\infty \le \varepsilon$. And $f \in C_0$ is trivial

Thm. $(C^{0,\alpha}(\bar{\mathbb{R}}), \|\cdot\|_{0,\alpha})$ is complete.

Pf. By above: (f_n) Cauchy in $C^{0,\alpha} \Rightarrow$ it's also Cauchy in C_0. So: $\exists f \in C$ st. $f_n \xrightarrow{u} f$

Same argument as above: $f_n \xrightarrow{\|\cdot\|_{0,\alpha}} f$.

Cor. (g_n) bdd in $C^{0,\beta}(\bar{\mathbb{R}}) \Rightarrow (g_n)$ has convergent subseq. in $C^{0,\alpha}(\bar{\mathbb{R}})$. $\forall \alpha < \beta$.

Pf: 1) Note $|g|_{0,\alpha} \le (2|g|_\infty)^{1-\frac{\alpha}{\beta}} |g|_{0,\beta}^{\alpha/\beta}$.

2) Apply Ascoli Thm.: (g_n) has

$\|\cdot\|_\infty$ - convergent subseq. $\to \tilde{g}$

let $g = g_n - \tilde{g}$ on 1)

Thm. $C_c^\infty \xrightarrow[*\varepsilon_n]{\text{approxi}} C_c \xrightarrow[\text{Urysohn}]{\text{approxi}}$ simple func. $\xrightarrow[\text{DCT}]{\text{approxi}} L^p$. $1 \le p < \infty$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad I_A$

Lem. $\mathcal{M}(\Omega)$ is set of $\mathbb{R}$-measures. Then:

$(\mathcal{M}(\Omega), \|\cdot\|_{TV})$ is Banach space.

Next, denote K is cpt set $\subset \Omega$.

Lem. $(\mathcal{M}(K), \|\cdot\|_{TV})$ isn't separable if $|K| > \aleph_0$

Pf: Set $\{\delta_x\}_{x \in K}$ is uncountable. And

$\|\delta_x - \delta_y\|_{TV} = 2$, $\forall x \ne y$.

Lem. $(C(K), \|\cdot\|_\infty)$ is separable.

Pf: $\{P_n^{(\alpha)}(x), n \in \mathbb{N}$. Weierstrass polynomials

with rational coeff$\}$ is dense.

Thm. $C(K)^* \cong \mathcal{M}(K)$ for K cpt.

Rmk: $(C_0(X))^* \cong (C_c(X))^* \cong \tilde{\mathcal{M}}(X)$.

$(C_B(X))^* \cong \tilde{\mathcal{M}}(X)$ for X is LCH

where $\tilde{\mathcal{M}}$ is set of Radon measures and $\widetilde{M}(X) = \widetilde{m}(X) \cap \{\mu$ is only finitely additive$\}$ via $\mu \mapsto (f \mapsto \int f d\mu)$

Rmk: i) For $(\mu_n) \subset M(X)$ bdd. Then:

$\mu_n \xrightarrow{*} \mu$ in $(C_c(X))^*$ is equi. with

$\mu_n \xrightarrow{*} \mu$ in $(C_0(X))^*$

Pf: By Banach – Steinhaus Thm.

Rmk: i) Note that weak topology is uniquely since E^* is unique

But weak* - topo isn't since

$\exists E \neq \widetilde{E}$. s.t. $E^* \cong \widetilde{E}^*$

ii) $(n\delta_n)$ isn't $\|\cdot\|_{TV}$ - bdd. let

$f(x) = \sin x / x \Rightarrow n\delta_n(f) = \sin(n)$

diverges. So: $n\delta_n \not\xrightarrow{*} 0$ in $C_0(X)^*$.

ii) $(\mu_n) \subset M(X)$ bdd. tight. Then:

$\mu_n \xrightarrow{*} \mu$ in $(C_0(X))^* \Leftrightarrow$ in $(C_B(X))^*$

e.g. (δ_n) on $f(x) = \sin(x) \in C_B \not\xrightarrow{*} 0$.

(δ_n will wander of to ∞)

Cor. $M(K)$ can't be reflexive if $|K| > \aleph_0$'

Pf: Note $c(k)$ is separable but $m(k)$ isn't. Contradict with $c(k)^* \cong m(k)$.

(3) Schwartz space:

Denote: $\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2}\int f(x) e^{-i\langle x,\xi\rangle} dx$. $\forall \xi$ for $f \in L^1(\mathbb{R}^n)$ is Fourier transform.

Thm: $\mathcal{F} \in \mathcal{L}(L^1(\mathbb{R}^n), C_0(\mathbb{R}^n))$. $\|\mathcal{F}\| \le (2\pi)^{-n/2}$.

Pf: 1) Conti. by DCT.

2) For $f \in C_c^\infty$. $\|\xi\|_2 \ge R$. $\exists |\xi_j| \ge \frac{R}{\sqrt{n}}$

$$|\mathcal{F}f(\xi)| = (2\pi)^{-n/2} \left|\int f(x) \frac{\partial}{\partial x_j} e^{-i\langle x,\xi\rangle} / \xi_j \right|$$

$$\le (2\pi)^{-n/2} \left\|\frac{\partial f}{\partial x_j}\right\|_{L^1} / |\xi_j|$$

$$\lesssim \sqrt{n}/R \to 0 \quad (R \to \infty).$$

And combine with $C_c^\infty \underset{L^1\text{-dense}}{\subseteq} L^1$.

3) $\|\mathcal{F}f\|_\infty \le (2\pi)^{-n/2}\|f\|_{L^1} \Rightarrow \mathcal{F}$ is BLO

Def: $\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \mid \partial^\beta f$ is rapid decreasing. i.e. $\lim\limits_{\|x\|_2 \to \infty} x^\alpha \partial^\beta f = 0 \}$

Lem. $\mathcal{F}$ will map $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. Besides,

$D^\alpha(\mathcal{F}f) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha f)$; $\mathcal{F}(D^\alpha f) = i^{|\alpha|}\xi^\alpha \mathcal{F}f$.

Pf: Note $\xi^\tau D^\beta \mathcal{F}f(\xi) = (-i)^{|\tau|+|\beta|}\mathcal{F}(D^\tau(x^\beta f))(\xi)$

Prop. For $f, g \in S(\mathbb{R}^n)$. We have:

i) $\mathcal{F}(\mathcal{F}f)(\xi) = f(-\xi)$. So: $\mathcal{F}^3 = \mathcal{F}^{-1}$.

ii) $\langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2} = \langle f, g\rangle_{L^2}$ (Parseval.)

Rmk: We can extend $\mathcal{F}: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ ($\subset L^2(\mathbb{R}^n)$) on $L^2(\mathbb{R}^n)$ which's also a isometric isomorphism: $\langle \mathcal{F}f, \mathcal{F}g\rangle = \langle f, g\rangle$, $\forall f, g \in L^2$ by density of $S(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$.

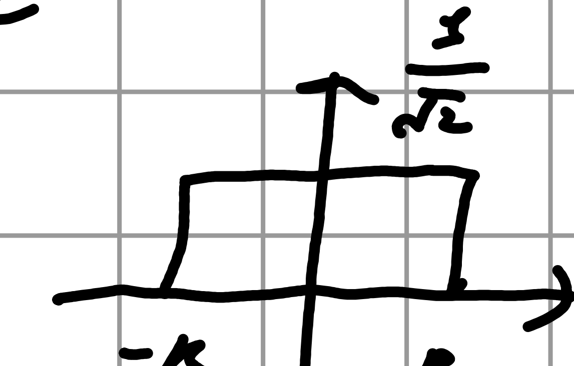
Thm. $H^m(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid (1+\|\xi\|^2)^{\frac{m}{2}}\mathcal{F}f(\xi) \in L^2(\mathbb{R}^n)\}$ and $\|f\|_{W^{m,2}} \sim \|(1+\|\xi\|^2)^{\frac{m}{2}}\mathcal{F}f(\xi)\|_{L^2}$.

Lem. $\mathcal{F}(e^{-|x|^2/2})(\xi) = e^{-|\xi|^2/2}$.

Pf: By residue formula: $f(x) = e^{-|x|^2/2}$.

$$\hat{f}(\xi) = (2\pi)^{-n} e^{-\frac{1}{2}|\xi|^2}\int e^{-\left|\frac{x}{\sqrt{2}} + i\frac{\xi}{\sqrt{2}}\right|_2^2}\,dx$$

Compute $\oint e^{-z_i^2}\,dz_i$ on



(let $R \to \infty$)

Lem. (Inversion Formula)

$$f(x) = (2\pi)^{-n/2} \int \tilde{f}(\xi) e^{ix\cdot\xi} d\xi.$$

Pf: By DCT: RHS $= \lim_{\varepsilon\to 0} (2\pi)^{-n/2} \int \hat{f}(\xi) e^{-\varepsilon^2|\xi|^2 + ix\xi} d\xi$

$$= \lim_{\varepsilon\to 0} \int f(y)\, \widehat{e^{\square}}(y)\, dy$$

with $\widehat{e^{\square}} = \varepsilon^{-n} \exp(-|x-y|^2/\varepsilon^2)$

So: RHS $= f(x)$.

(4) Quotient Space:

Def: "$\sim$" relation is equivalence on set X if a) $x \sim x$ b) $x \sim y \iff y \sim x$.

c) $x \sim y,\ y \sim z \Rightarrow x \sim z$. $\forall x, y, z \in X$.

Lem. $(E, \|\cdot\|)$ n.v.s. $F \subset E$ closed subspace.

Then: $\|[x]\| = \inf_{y\in F} \|x+y\|$ defines a norm on E/F. And E/F is Banach if E is

Pf: We need closedness for $\|[x]\| = 0 \iff x \in F$. $(\exists y_n \in F,\ x + y_n \to 0. \Rightarrow x = \lim -y_n \in F)$

For the latter: $\exists (x_{n_k}).\ (y_k)$.

$\|x_{n_k} - x_{n_{k-1}} + y_{k-1}\| < 2^{-k}$. let $\eta_k = x_{n_k} + z_k$.

$z_k = z_{k-1} + y_{k-1}$. So: (η_k) is Cauchy in E.

$\Rightarrow \eta_k \to \eta$. With $\|[x_{n_k}] - [\eta]\| \le \|\eta_k - \eta\| \to 0$

<u>Lem.</u> E n.v.s. $F \subset E$. CLS. Then:

E is Banach $\Leftrightarrow$ F. E/F are Banach.

<u>Pf:</u> We've proved $(\Rightarrow)$. For $(\Leftarrow)$:

(x_n) Cauchy in E $\Rightarrow$ $([x_n])$ is Cauchy in E/F. $\Rightarrow [x_n] \to [x]$.

$\Rightarrow \exists (y_n)$. $\|x_n - x + y_n\| \to 0$. $\Rightarrow (y_n)$ is Cauchy in $F \to y$. $\Rightarrow x_n \to x - y$.

<u>Def:</u> $T: E \to F$ is quotient map if that:

$T(\{x \in E \mid \|x\| < 1\}) = \{y \in F \mid \|y\| < 1\}$.

<u>Lem.</u> linear quotient map T is surjective with $\|T\| = 1$

<u>Pf:</u> i) $\forall y \in F$. $\exists x$. $Tx = \frac{y}{2\|y\|} \Rightarrow T(2\|y\|x) = y$.

2) $\|T(c\frac{x}{\|x\|})\| < 1$. $\forall c \in (0,1)$. So:

$\|Tx\| \le c\|x\| \xrightarrow{c \to 1} \|x\|$.

And for $y_0 \in F$. $\|y_0\| = 1$. $\exists x_c \in E$. s.t.

$T(x_c) = cy_0 \Rightarrow \|T(x_c)\| = c \to 1$.

Rmk: More generally, any open linear op. is surjective.

Pf: Since $\exists B_r^E(0) \subset R(T) \subset F$

Rmk: Open linear operator $T: E \to F$ may not be closed. e.g.

$l: (x_n) \in C_0 \mapsto \sum 2^{-n} x_n \in \mathbb{R}^1$.

$\Rightarrow |l(x)| \le \sum 2^{-n} |x_n| \le \sum 2^{-n}$

for $\forall x \in C_0$. $\Rightarrow$ l is bdd.

So by open mapping, l is open

But l isn't closed, since

$|l((I_{\{n \le N\}})_n)| = \sum_1^N 2^{-n} \to 1 \notin l(\bar{B}_1)$

Lem. $F \subset E$ closed subspace of n.v.s. Then:

$w: x \in E \to [x] \in E/F$ is quotient map.

Pf: For $\forall x$, $\|[x]\| < 1$. $\Rightarrow \exists y \in F$. s.t.

$\|x+y\| < 1$. So: $w(x+y) = [x]$.

$\forall x$. $\|x\| < 1$. $\Rightarrow \|[x]\| \le \|x\| < 1$.

So: $w(B_E) = B_{E/F}$.

Thm. $\exists$ unique BLO $\tilde{T}: E/\ker(T) \to F$. for T

$\in \mathcal{L}(E, F)$. s.t.
$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "w"'] & F \\ E/\ker(T) \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$
commutes. where $w: x \mapsto [x]$.

Then: i) $\hat{T}$ is injective. $\|\hat{T}\| = \|T\|$.

ii) If T is quotient map. Then:

$\hat{T}$ is isometric isomorphism.

<u>Rmk:</u> $\hat{T}$ may not be isomorphism. <u>e.g.</u> $E = \{(x_n) \mid n^k x_n \to 0, \forall k\}$ with $\|\cdot\|_\infty$. $T: (x_n) \in E \mapsto (x_n/n) \in E \Rightarrow \|T\| \le 1$.

But $T^{-1} = S(x_n) = (nx_n)$ isn't bdd. $\Rightarrow$ $\|\hat{T}^{-1}\| = \|T^{-1}\|$ not bdd.

<u>Pf:</u> i) Set $\hat{T}[x] := Tx$. Then:

$\hat{T}$ is well-def., linear and injective

uniqueness is by: $\hat{T} \circ w = T$. $R(w) = E/\ker T$.

For boundness: Note $\exists y_n \in \ker T$. $\|x + y_n\| \to \|[x]\|$

So: $\|\hat{T}[x]\| = \|Tx\| = \|T(x+y_n)\| \le \|T\| \|x + y_n\|$

$\to \|T\| \|[x]\|$.

Also: $\|Tx\| = \|\hat{T}[x]\| \le \|\hat{T}\| \|[x]\| \le \|\hat{T}\| \|x\|$

$\Rightarrow \|T\| = \|\hat{T}\|$

ii) T is quotient map $\Rightarrow \|T\| = \|\tilde{T}\| = 1$.

Also $T = \tilde{T} \circ \omega$ implies $\tilde{T}$ is surjective.

Note T is quotient. Then $\forall y \in F$. $\|y\| < 1$.

$\exists x$. s.t. $Tx = y = \tilde{T}[x]$. $\|[x]\| \leq \|x\| < 1$.

So: $\|\tilde{T}^{-1} y\| < 1$. i.e. $\|\tilde{T}^{-1}\| \leq 1$.

$\Rightarrow \tilde{T}$ is isomorphism.

__Thm__: $F \subseteq E^*$. subspace of n.v.s. Then:

i) $q: E^*/F^\perp \to F^*$. $q(f + F^\perp) = f|_F$. $f \in E^*$

is isometric isomorphism.

ii) For F closed. $q: (E/F)^* \to F^\perp$ defined

by $qf(x) := f(x+F)$. $x \in E$. $f \in (E/F)^*$

is isometric isomorphism.

__Pf__: i) Consider $T: f \in E^* \mapsto f|_F \in F^*$ is linear

well-def. surjective and $\ker(T) = F^\perp$

For q is isometric:

Note $\|q(f + F^\perp)\| = \|f|_F\| = \|(f+g)|_F\|$

$\leq \|f+g\|$. $\forall g \in F^\perp$

let $\tilde{f} \in E^*$. s.t. $\tilde{f}|_F = f|_F$. $\|\tilde{f}\| = \|f|_F\|$

$\Rightarrow \|f + F^\perp\| \le \|f + (\tilde{f} - f)\| = \|f|_F\|$

$= \|\phi(f + F^\perp)\|$

ii) Check ϕ is well-def. linear. inject

For surjective: $\forall g \in F^\perp$. Set f by

$f(x + F) = g(x)$. $\forall x \in E \Rightarrow f \in (E/F)^*$

For isometric: $\|\phi f(x)\| \le \|f\| \|x\|$.

Conversely, find $x \in E$. $|f(x+F)| > \|f\| - \varepsilon$

and $y \in F$. $\|x + F\| + \varepsilon \ge \|x + y\|$.

Consider $x+y / \|x+F\| + \varepsilon \Rightarrow \|\phi f\| \ge \|f\| - \varepsilon$.

Lem. E^{lk} is n.v.s. Then:

i) E reflexive. $F \subset E$ CLS $\Rightarrow F$ reflexive

ii) E Banach. Then: E reflexive $\Leftrightarrow E^*$ is

Pf: i) $\forall \varphi \in F^{**}$. Set $\psi : l \in E^* \mapsto \varphi(l|_F)$

$\Rightarrow \psi \in E^{**}$. $\exists y \in E$. $\psi(l) = l(y)$.

Next, prove: $y \in F$ by contradiction

and Hahn-Banach ($\exists l|_F = 0$. $l(y) \ne 0$)

ii) $(\Rightarrow)$. For $u \in E^{***}$. $\forall \varphi \in E^{**}$. $\exists \hat{x} = \varphi$

Set $f(x) = u(\hat{x})$. $f \in E^*$. $\Rightarrow u(\varphi) = \varphi(f)$,

$(\Leftarrow)$ By $(\Rightarrow)$. So: E^{**} reflexive and

$\Lambda_E(E) = \overline{\Lambda_E(E)} \cong E$ with i)

<u>Thm.</u> E Banach. $F \subset E$. CLS. Then:

E reflexive $\Leftrightarrow$ F, E/F reflexive.

<u>Pf:</u> $(\Rightarrow)$ $(E/F)^* \cong F^\perp \leq E^*$ reflexive.

$(\Leftarrow)$ For $\varphi \in E^{**}$. we can find x, q

from ref of E/F and F. s.t. $\varphi = \widehat{x+q}$

Denote $\Phi : (E/F)^* \cong F^\perp$ above.

Construct $\psi \in (E/F)^{**}$. define by

$\psi(u) = \varphi(\Phi(u))$. $u \in (E/F)^*$. $\Rightarrow \exists \psi = \widehat{x+F}$

and check $(\varphi - \hat{x})|_{F^\perp} = 0$.

We set $\ell(f) := (\varphi - \hat{x})(g)$, $f \in F^*$ where

$g|_F = f$. $\|g\| = \|f\|$. $g \in E^*$.

$\Rightarrow \ell \in F^{**}$ is well-def. $\exists \ell = \hat{q}$.

So: $\varphi = \widehat{x+q}$.