Linear Operators

(1) Examples:

① $T: (\mathbb{K}^n, \|\cdot\|_1) \to (\mathbb{K}^m, \|\cdot\|_1)$. $T = (T_{ij})_{m\times n}$

$$\Rightarrow \|T\|_* = \max_{1\le j\le n} \sum_{i=1}^m |T_{ij}|.$$

Pf: $\|Te_j\|_1 = \sum_{i=1}^m |T_{ij}| \le \|T\|_* \Rightarrow RHS \le \|T\|_*$.

$$\|Tx\|_1 = \sum_{i=1}^m |(Tx)_i| \le \sum_j \sum_i |T_{ij}||x_j|$$

$$\le \max_{1\le j\le n} \sum_{i=1}^m |T_{ij}| \cdot \|x\|_1$$

② $T: (\mathbb{K}^n, \|\cdot\|_\infty) \to (\mathbb{K}^m, \|\cdot\|_\infty)$. $T = (T_{ij})_{m\times n}$

$$\Rightarrow \|T\|_* = \max_{1\le i\le m} \sum_{j=1}^n |T_{ij}|.$$

Pf: $\|T(\operatorname{sgn}(T_{ij}))_j\|_\infty = \sum_{j=1}^n |T_{ij}| \le \|T\|_*$.

$$\|Tx\|_\infty \le \max_i \Big|\sum_j T_{ij}x_j\Big| \le \max_i \sum_{j=1}^n |T_{ij}| \cdot \|x\|_\infty.$$

③ $T: (\mathbb{K}^n, \|\cdot\|_2) \to (\mathbb{K}^m, \|\cdot\|_2)$. $T = (T_{ij})_{m\times n}$

$$\Rightarrow \|T\|_* = \lambda_{max}(T^T T).$$

Pf: $\|Tx\|_2^2 = x^T T^T T x$

④ $T: C([0,1];\mathbb{K}) \to \mathbb{K}$. $x \mapsto x(0) \Rightarrow \|T\|_* = 1$.

Pf: $\|T\|_* = 1$. And $|T(I_{[0,1]})| = 1$.

(5) $T_g : C([0,1]) \to \mathbb{K}$. $T_g x = \int_0^1 x(t) g(t) dt$.

$\Rightarrow \|T_g\|_* = \|g\|_{L^1}$.

Pf: $|T_g(x)| \leq \|x\|_\infty \|g\|_{L^1}$. And let $x_\varepsilon = \frac{\bar{g}}{|g|+\varepsilon}$

$\Rightarrow |T_g(x_\varepsilon)| \geq \|g\|_{L^1} - \varepsilon$. let $\varepsilon \to 0$.

(2) Basic Lem.:

Lem. E, F. n.v.s. $\dim E < \infty$. $L : E \to F$. linear.

Then: i) All norms in E are equi.

ii) L is conti.

Pf: i) Set $(x_i)_1^n$ basis of E. Define:

$\|x\|_1 = \|\sum \lambda_i(x) x_i\|_1 := \sum |\lambda_i(x)|$

$\Rightarrow \|x\|_E \leq \max \|x_i\|_E \|x\|_1$

Conversely. $(E, \|\cdot\|_1) \xrightarrow{id} (E, \|\cdot\|_E)$

$\xrightarrow{\|\cdot\|_E} \mathbb{K}$ is conti. S_E is cpt.

So: $\|\cdot\|_E \circ id$ attain $\min_{\|x\|_1=1} \square = c > 0$

$\Rightarrow \|x\|_E / \|x\|_1 > c$.

ii) $\|Tx\|_F \leq \max \|T(x_i)\|_F \|x\|_1 \lesssim \|x\|_E$

Cor. E is complete.

Pf: $x_i \in E \mapsto e_i \in \mathbb{R}^n$ is isomorphism

Lem. E, F, G are n.v.s. $T \in \mathcal{L}(E,F)$. $S \in \mathcal{L}(F,G)$

$\Rightarrow \|S\circ T\| \leq \|S\|\|T\|$.

Rmk: It's not optimal. ($\exists$ $\|S\circ T\| < \|S\|\|T\|$)

e.g. i) (unbdd D) $Du = u'$ on $u \in C([0,1], \|\cdot\|_\infty)$

ii) $T: (x_n) \in \ell_\infty \mapsto (x_n/n) \in \ell_\infty$ has unbdd inverse $T^{-1}(x_n) = (n x_n)$

Lem. E, F n.v.s. Then: $\mathcal{L}(E,F)$ is Banach

$\Leftrightarrow$ F is Banach.

Pf: ($\Leftarrow$) (T_n) Cauchy implies $(T_n x)$ Cauchy.

So: $T_n x \to \tilde{x} =: Tx$. $T \in \mathcal{L}(E,F)$

($\Rightarrow$) (y_n) Cauchy in F.

$\exists\, \ell \in E^*$. s.t. $|\ell(x_0)| = \|x_0\| = 1$.

Set $A^n x := \ell(x) y_n \in \mathcal{L}(E,F)$

$\Rightarrow A^n$ is Cauchy.

Lem. E, F. $\mathbb{K}$-n.v.s. $T: E \to F$ is linear.

Thm: $\exists B \in \mathcal{L}: T^{-1}: R(T) \to E \iff$

$\exists c > 0.\ \|x\| \leq c\|Tx\| \overset{T \in \mathcal{L}}{\iff}$

if E.F. Banach

$N(T) = \{0\}$. $R(T)$ is closed.

<u>Lem.</u> i) φ is conti. $\iff$ $\ker(\varphi)$ is closed.

ii) φ isn't conti. $\iff$ $\ker(\varphi) \subsetneq E$. dense.

<u>Pf:</u> i) ($\Rightarrow$) is trivial. For ii) ($\Rightarrow$):

$\exists (x_n) \to 0$. But $\varphi(x_n) \not\to 0$. i.e.

$\exists (x_{n_k}) \to 0$. and $|\varphi(x_{n_k})| > c > 0$.

Set $y_k = x_{n_k} / \varphi(x_{n_k}) \Rightarrow \|y_k\| \to 0$.

Note $\forall x \in E$. $z_k = x - \varphi(x) y_k \in \ker(\varphi)$.

And $z_k \to x$. So $\ker(\varphi)$ is dense.

So: i) ($\Leftarrow$) and ii) ($\Leftarrow$) are true

(3) <u>Basic Thm:</u>

① <u>Hahn Banach:</u>

<u>Def:</u> "$\leq$" is partial order on set A if:

a) $x \leq x$. $\forall x \in A$. b) $x \leq y$. $y \leq x \Rightarrow x = y$.

c) $x \le y,\ y \le z \Rightarrow x \le z$

$m \in A$ is max if $\forall x \ge m \Rightarrow x = m$.

$s \in K \subset A$ is upper bdd if $s \ge x,\ \forall x \in K$.

And $K \subset A$ is chain if "$\le$" on K is a total order

e.g. $(P(X), \le)$ is a poset.

Lem. (Zorn's)

In poset $(A, \le)$, If $\forall$ chain K has a upper bdd. Then: A has max element.

Thm. (Hahn - Banach)

Only require $F \subset E^{\mathbb{k}}$, LS. And $f \in F'$ (only algebraic dual space). $f(x) \le e(x)$ on F, where e is sublin ($\begin{matrix} e(x+y) \le e(x) + e(y) \\ e(\lambda x) = \lambda e(x),\ \lambda > 0 \end{matrix}$)

$\Rightarrow \exists \tilde{f} \in E',\ \tilde{f}|_F = f,\ \tilde{f} \le e$ on E.

Cor. For e is Seminorm ($\begin{matrix} e(x+y) \le e(x) + e(y) \\ e(\lambda x) = |\lambda| e(x),\ \forall \lambda \in \mathbb{k} \end{matrix}$) and $|f(x)| \le e(x)$ on F. Then: the extension $|\tilde{f}(x)| \le e(x)$ on E.

__Pf:__ $\mathbb{K} = \mathbb{R}$: $-\tilde{f}(x) = \tilde{f}(-x) \le \ell(-x) = \ell(x)$.

$\mathbb{K} = \mathbb{C}$: By linearity in $\mathbb{C}$. $f(ix) = i f(x)$.

Assume $f(x) = f_1(x) + i f_2(x)$. $\Rightarrow$

$f_2(x) = -f_1(ix)$. i.e. $f(x) = f_1(x) - i f_1(ix)$

Apply Thm on $f_1(x)$. $\exists\ \tilde{f}_1$ extend it

Set $\tilde{f}(x) = \tilde{f}_1(x) - i \tilde{f}_1(ix) \Rightarrow \mathbb{C}$-linear

And $|\tilde{f}(x)|^2 = \bar{\tilde{f}} \tilde{f} = \tilde{f}(\bar{\tilde{f}}(x) x)$

$= \tilde{f}_1(\bar{\tilde{f}}(x) x)$ (LHS $\in \mathbb{R}$)

$\le \ell(\bar{\tilde{f}}(x) x) = |\tilde{f}(x)| \ell(x)$

__Cor.__ For $F \subset E$ n.v.s. $\Rightarrow \forall\, \ell \in F^*$. $\exists\, L \in E^*$

s.t. $L|_F = \ell$. $\|\ell\| = \|L\|$.

__Pf:__ Set $\ell(x) = \|\ell\| \|x\|$. seminorm.

$\Rightarrow \|L\| \le \|\ell\|$. with $L|_F = \ell \Rightarrow \|L\| = \|\ell\|$.

__Rmk:__ For $F \subset E$ dense. By continuity extension. L is unique. (Also for G Banach. $\ell \in \mathcal{L}(F, G)$ can also be uniquely extended to $\mathcal{L}(E, G)$)

Pf: $(x_n) \in F \to x \in E$. $\Rightarrow$ (Tx_n) is

Cauchy $\to \widehat{Tx}$ since G is Banach

But generally, $\mathcal{L} = \{ L \in E^* \mid L|_F = \ell$

$\|L\| = \|\ell\| \}$ may not be singleton.

i) $\mathcal{L}$ is convex (easy to check)

ii) If E^* is strictly convex (i.e. $\|L_1\| = \|L_2\| = 1 \Rightarrow \|\frac{L_1+L_2}{2}\| < 1$. $\forall L_1 \neq L_2 \in E^*$)

Then: $\# \mathcal{L} = 1$.

Pf: If $\exists L_1 \neq L_2 \in \mathcal{L} \Rightarrow \|\frac{1}{2}(\frac{L_1}{\|\ell\|} + \frac{L_2}{\|\ell\|})\|$

$< 1 \Rightarrow \|L_1 + L_2\|/2 < \|\ell\|$.

But $L_1 + L_2/2$ extend ℓ. So:

$\|L_1 + L_2\|/2 \geq \|\ell\|$. Contradict!

e.g. ℓ_p is strictly convex. $\forall 1 < p < \infty$

(By Minkowski: $\|y_1 + y_2\|_q < \|y_1\|_q + \|y_2\|_q$

for $y_1 \neq y_2$. Since y_1, y_2, l.i.)

So for $E = \{e_1, \cdots e_n\} \subset \ell_p$.

Since $E^* = \{ x \mapsto f_y(x) = \sum_{i}^{n} x_i y_i \}$.

$l \in E^*$. s.t. $l(x) = \sum_1^n x_i y_i$. its extension $L \in l_p^*$. s.t. $L|_E$. $\|L\| = \|l\|$ is uniquely define by $L(x) = \sum_1^\infty x_i y_i$.

e.g. (Momentum Problem)

$(a_n) \subseteq \mathbb{R}'$. Then: $\exists \mu \in M(K)$. s.t. $a_n = \int t^n d\mu$

$K \subseteq \mathbb{R}'$. cpt $\iff \exists c > 0$. $\forall n$. $\forall \lambda_i$. $|\sum_0^n \lambda_i a_i| \le c \|\sum_1^n \lambda_i t^i\|_{C(K)}$

Pf: $t^i \in C(K)$. Note $C(K)^* \cong M(K)$.

By Hahn-Banach. $(\Leftarrow)$ $l(x) = c\|x\|$ and $(\Rightarrow)$ $c = \|l\|$.

Rmk: It's criteria that whether we can find p.m. μ. s.t. $E^\mu(X^k) = a_k$.

Cor. i) $\exists l \in E^*$. s.t. $|l(x)| = \|x\|$. $\|l\| = 1$.

CS. : $\forall x_1 \neq x_2$. $\exists l \in E^*$. $l(x_1) \neq l(x_2)$

Pf: extend on $\mathrm{Span}\{X\}$. (let $X = x_1 - x_2$)

ii) $F \subset E$. subspace of n.v.s. $x \in E$. s.t. $d(x, F) > 0$. $\Rightarrow \exists l \in E^*$. $l|_F = 0$. $\|l\| = 1$

and $\ell(x) = d(x, F)$.

Pf: Extend $g(y+\lambda x) = \lambda\delta$. $y \in F$ on $G := F + \mathbb{K}x$. $\Rightarrow \|g\| = 1$.

Cr. F is dense $\Leftrightarrow \forall \ell \in E^*$. $\ell|_F = 0$ implies $\ell = 0$.

② Open mapping:

Lem. E n.v.s. F Banach $T \in \mathcal{L}(E, F)$ surjective $\Rightarrow \exists r > 0$. s.t. $B_1^F(0) \subset \overline{T(B_r^E(0))}$

Pf: F Banach is for applying Baire thm on $F = \bigcup \overline{T(B_n^E(0))}$

Lem. E Banach. F n.v.s. $T \in \mathcal{L}(E, F)$. st. $\exists r > 0$ $B_1^F(0) \subseteq \overline{T(B_r^E(0))} \Rightarrow B_1^F(0) \subset T(B_{2r}^E(0))$ and T is surjective.

Pf: Fix $y \in B_1^F(0)$. Construct $(y_n) \subset T(B_r^E(0))$ st. $\|y - \sum_1^n y_k/2^{k-1}\| \le 2^{-n}$ by induction.

So: $\exists (x_n) \subset B_r^E(0)$. st. $Tx_n = y_n$.

And $\left(\sum_1^n \frac{x_k}{2^{k-1}}\right)$ is Cauchy $\to x \in B_{2r}^E(0)$

$\Rightarrow Tx = T(\lim \sum \frac{x_n}{2^{n-1}}) = \lim \sum \frac{y_n}{2^{n-1}} = y$.

So: open mapping Thm requires E, F are both Banach to apply the two Lemmas.

<u>Lem.</u> Set (b_n) is Schauder basis in Banach space E. Then: $b_n^* : x = \sum_1^\infty \lambda_n b_n \in E \mapsto \lambda_n \in \mathbb{K}$ is linear, conti. $\forall n$.

<u>Pf</u>: Set $|||x||| = \sup_N \| \sum_1^N b_n^*(x) b_n \|$

So: $|b_n^*(x)| = \| \sum_1^n b_k^*(x) b_k - \sum_1^{n-1} b_k^*(x) b_k \|$

$$\leq 2\, |||x|||$$

Next, we prove $|||\cdot||| \sim \|\cdot\|$ in E.

$(E, |||\cdot|||) \xrightarrow{id} (E, \|\cdot\|)$. is BLO by:

$\|x\| = \lim \| \sum_1^N b_n^*(x) b_n \| \leq |||x|||$.

To apply open mapping Thm, Next we show $(E, |||\cdot|||)$ is Banach space.

For (x^n) is $|||\cdot|||$ - Cauchy seq.

$|b_n^*(x^k) - b_n^*(x^l)| = \| b_n^*(x^k - x^l) b_n \|$

$$\leq 2\, |||x^k - x^l|||.$$

So: $b_n^*(X^k) \xrightarrow{k\to\infty} \lambda_n$. $\Rightarrow \exists N$ $\| \sum_{1}^{N} b_j^*(X^k) b_j - \lambda_k b_k \| < \varepsilon$

$\Rightarrow \| \sum_{N_1}^{N_2} \lambda_k b_k \| \leq 2\varepsilon + \| \sum_{N_1}^{N_2} b_j^*(X^k) b_j \|$.

$\leq 3\varepsilon$. $\forall N_1, N_2$ large enough

So: $\sum_1^n \lambda_k b_k$ converge in $(E, \|\cdot\|) \to X$

Similarly we have $|||X^k - X||| \to 0$.

e.g. (E, F Banach is necessary)

$T: (C[0,1], \|\cdot\|_\infty) \to (\{ C^1[0,1] : g(0)=0 \}, \|\cdot\|_\infty)$

$f \mapsto \int_0^\cdot f \, dx$

is bdd. surjective ($T(g') = g \in C^1$). But

$T^{-1}u = u'$ isn't bdd. ($T^{-1}t^n = n \to \infty$)

⑧ Closed Graph Thm:

i) Require E, F are Banach (to apply open mapping Thm.)

ii) T is linear operator is necessary.

e.g. $f(x) = x^{-1} I_{\{x>0\}}$ on $\mathbb{R}^1$ has closed graph. But it's not conti.

Lem. $F < (E, \|\cdot\|)$ subspace of Banach space. If $(F, \|\cdot\|_1), (F, \|\cdot\|_2)$ are both Banach $\xrightarrow{id_1, id_2} E$ is conti. embedding $\Rightarrow \|\cdot\|_1 \sim \|\cdot\|_2$

Pf: Prove: $(F, \|\cdot\|_1) \xrightarrow{I} (F, \|\cdot\|_2)$ is isomorphism.

Check by CGT: $(x_n) \to x$ in $\|\cdot\|_1$ and $x_n \to y$ in $\|\cdot\|_2$. Since id_1, id_2 are conti. $\Rightarrow x = y$. So I is closed

④ **Banach – Steinhaus Thm:**

· UBP require $\mathcal{T} \subset \mathcal{L}(E, F)$, E is Banach

Since we need to apply it on $E = \bigcup_n \{x \mid \|Tx\| \le n, \forall T \in \mathcal{T}\}$.

e.g. (Banach is necessary)

$f_n : c_{00} \ni x \mapsto n x_n \in \mathbb{K}'$. (f_n) is pointwise bdd but not uniformly.

Lem. E, F n.v.s. $(T_n) \in \mathcal{L}(E, F) \to T$ pointwise. (T_n) is uniform bdd $\Rightarrow T \in \mathcal{L}(E, F)$

Pf: $\|Tx\| = \lim_n \|T_n x\| \le \sup \|T_n\| \cdot \|x\|$

Cor. For E Banach. $T_n \in \mathcal{L}(E,F) \xrightarrow{\text{ptwise}} T$

$\Rightarrow T \in \mathcal{L}(E,F)$. $\|T\| \le \underline{\lim} \|T_n\|$

Pf: (T_n) is pointwise bdd. Apply

UBP $\Rightarrow (\|T_n\|)$ is bdd.

Choose $\|T_{n_k}\| \to \underline{\lim} \|T_n\|$.

So: $\|Tx\| = \lim \|T_{n_k} x\| \le \underline{\lim} \|T_n\| \|x\|$

e.g. ptwise convergence $\overset{\not\to}{\leftarrow}$ Unif. conv.

$f_n : x \in c_0 \mapsto x_n \in \mathbb{K} \to 0$ pointwise.

But $\|f_n\| = 1 \not\to 0$.

Thm. (Banach - Steinhaus)

E, F Banach. Then: $(T_n) \subset \mathcal{L}(E,F) \to$

$T \in \mathcal{L}(E,F)$ ptwise $\Leftrightarrow$ i) $\sup \|T_n\| \le M$

ii) $\exists D \subset E$ dense. $(T_n y)$ converge $\forall y \in D$.

Pf: $(\Rightarrow)$ is Cor. above. For $(\Leftarrow)$

We need $\|T_n x - T_m x\| \le \|T_n\| \|x - y\|$

$+ \|T_m\| \|x - y\| + \|T_n y - T_m y\|$. $y \in D$.

$\Rightarrow (T_n x)$ is Cauchy in F.

Cor. (Num. Quadrature)

Set $I : f \in C[a,b] \mapsto \int_a^b f \in \mathbb{R}'$ and

$I_n : f \in C[a,b] \mapsto \sum_j^n w_j^{(n)} f(x_j^{(n)})$. Then:

$I_n \to I$ pointwise $\iff$

i) $I_n(p) \to I(p)$. $\forall p$ polynomial.

ii) $\sup_n \sum_i^n |w_i| \le M < \infty$.

Pf: Note polynomials is dense in

$C(C[a,b], \|\cdot\|_\infty)$. Besides:

$\|I_n f\| \le \sum_i^n |w_i^{(n)}| \, \|f\|_\infty$.

And set $f(x_j^{(n)}) = \mathrm{sgn}(w_j^{(n)}) \|f\|_\infty$

$\Rightarrow \|I_n\| = \sum_i^n |w_i^{(n)}|$.

e.g. For $1 \le p < \infty$, $A : f \in l_p \mapsto (Af(n)) \in l_p$

$Af(n) := \sum_{k=1}^n \alpha_{nk} f(k) \Rightarrow A \in \mathcal{L}(l_p)$.

Pf: Set $l_n^{(i)} : f \in l_p \mapsto \sum_j^n \alpha_{ij} f_j \in \mathbb{R}'$.

By assumption: $Af(n)$ is finite

$\Rightarrow l_n^{(i)} \to A(\cdot)(i)$ pointwise.

So: $A(\cdot)(i) \in l_p^*$. $(\alpha_{ij})_j \in l_q$. And

$\| Af(i) \| \le \| (\alpha_{ij})_j \|_q \, \| f \|_p$.

$\Rightarrow A_n f = (Af(1), \cdots Af(n), 0, 0 \cdots) \in \mathcal{L}(\ell_p)$

and $A_n f \xrightarrow{pt} Af$ by cond. $\Rightarrow A \in \mathcal{L}(\ell_p)$

<u>Cor.</u> (X, Ω, μ) is σ-finite. $1 \le p < \infty$. If

$k(x, \cdot) f(\cdot) \in L^1(\mu)$ and $Kf(x) = \int k(x,y) f(y) d\mu(y) \in L^p(\mu)$. $\forall f \in L^p$. $\Rightarrow K \in \mathcal{L}(L^p(\mu))$

<u>Pf</u>: Set $\Omega_n^* = \Omega_n \cap [k(x, \cdot) < n]$. s.t.

$\Omega = \bigcup \Omega_n$. $\mu(\Omega_n) < \infty$.

$K_n^* f = \int_{\Omega_n^*} k(x,y) f(y) d\mu(y) \in L_p^* \xrightarrow{pt} Kf(x)$

So : $Kf(x) \in L_p^* = L_q$

Set $D_n = [\| k(x, \cdot) \|_{L_q} < n] \cap \Omega_n$ and

$T_n f(x) = I_{B_n} \cdot \int k(x,y) f(y) d\mu(y) \in \mathcal{L}(L_p)$

$\xrightarrow{pt} Tf \in \mathcal{L}(L_p)$

④ <u>Separation Thm.</u>

<u>Thm.</u> For E n.v.s.

i) $M \subset E$. closed convex. $x_0 \notin M$. $\Rightarrow$

$\exists \ell \in E^*$. s.t. $\mathrm{Re} \langle x, \ell \rangle \le \alpha < \mathrm{Re} \langle x_0, \ell \rangle$ $\forall x \in M$.

ii) $M_1, M_2 \subset E$ convex. $M_1 \cap M_2 = \phi$.

if a) M_2 open

or b) M_1 closed, M_2 cpt.

then: $\exists\, l \in E^*$. st. for $\forall x_1 \in M_1,\ x_2 \in M_2$

$Re\langle x_1, l\rangle \le a < Re\langle x_2, l\rangle$

Pf: i) For $\mathbb{K} = \mathbb{R}'$: Apply Hahn-Banach on

$f(\lambda x_0) = \lambda P_M(x_0)$ on $span\{x_0\}$. st.

P_M is gauge func. to $l \in E^*$.

For $\mathbb{K} = \mathbb{C}$: set $\tilde{l}(x) = l(x) - i\, l(ix)$

ii) Similarly, we prove case $\mathbb{K} = \mathbb{R}'$. and

Let $\tilde{l}(x) = l(x) - i\, l(ix)$ in $\mathbb{K} = \mathbb{C}$.

a) let $M = M_1 - M_2 = \bigcup_{M_1} (\{x\} - M_2)$ open

and $0 \notin M$. So by i): $\exists\, l \in E^*$. st.

$l(x_1 - x_2) = l(x_1) - l(x_2) < l(0) = 0 \quad \forall x_i \in M_i$

$\Rightarrow l(x_1) \le \inf_{M_2} l(x_2) \le l(x_2)$.

Next prove: $\inf_{M_2} l(x_2)$ can't attain

otherwise. $\exists\, x_0 \in M_2$. $l(x_0) = \inf_{M_2} \square$

$\Rightarrow \exists\, B_r(x_0) \subset M_2$

If $\ell \equiv c$ on $B_r(x_0) \Rightarrow B_r(0) \subset \ker(\ell)$

So: $\ell \equiv 0$. $\Rightarrow \exists\, x_0 + y \in B_r(x_0)$. s.t.

$\ell(x_0+y) \neq \ell(x_0) \Rightarrow \ell(y) \neq 0$. So:

$\ell(x_0+y) \wedge \ell(x_0-y) < \ell(x_0)$. Contradiction

<u>Rmk</u>: Linear functional can't attain its inf / sup on open set.

b) Check $M_1 - M_2$ is closed and convex

Then $\exists\, B_r(0) \cap (M_1 - M_2) = \emptyset$.

$\Rightarrow$ reduce b) to i).