Reflexive.

① For $x_n \to x$. $f_n \xrightarrow{*} f \Rightarrow f_n(x_n) \to f(x)$. We require $(x_n) \subset E$ Banach

<u>Pf</u>: $|f_n(x_n) - f(x)| \leq \|f_n\| \|x_n - x\| + |f_n(x) - f(x)|$

Note we use UBP to prove $(\|f_n\|)$ bdd.

<u>Rmk</u>: i) $x_n \rightharpoonup x$. $f_n \to f \Rightarrow f_n(x_n) \to f(x)$ has no require on E. (Since E^* Banach)

ii) <u>e.g.</u>, $E = c_{00}$ not Banach.

Consider $\langle \cdot, n e^n \rangle \in c_{00}^* \rightharpoonup 0$ and $(x^n) \subset c_{00}$. $x_k^n = n^{-1} I_{\{k=n\}}$. $\Rightarrow x^n \to 0$

But $\langle x^n, n e^n \rangle = 1$.

iii) $x^n \rightharpoonup x$. $f_n \rightharpoonup f \not\Rightarrow f_n(x^n) \to f(x)$.

<u>e.g.</u>, $(x^n) = (e^n)$. $(f_n) = (\langle \cdot, e^n \rangle)$

on $E = F = \ell_2$.

② <u>Thm</u>. E separable n.v.s. $\Rightarrow E^*$ is weak*-seq. cpt.

Pf: (x_n) is dense in E. $(f_n) \subset E^*$. (bdd)

S.: $(f_n(x_1))_n$ has conv. subseq. $(f_{n,1})$. Also for $(f_{n,1}(x_2))$...

By diagonal argument. $\exists (f_{n,n}) \subset (f_n)$ converges on (x_n). $\Rightarrow$ extend on E.

e.g. (Sep. is necessary).

$E = l_\infty$. Set $f_n(x) = x_n$. $\Rightarrow \|f_n\| = 1$.

If $\exists (f_{n_k})$ converge subseq. we let

$x = (x_n) := (I_{\{n=n_k, k \text{ is even}\}})$.

But $f_{n_k}(x)$ doesn't converge.

Rmk: $\bar{B}_1^{E^*}$ is weak*-cpt in E^* as well

Cor. E reflexive Banach. $\Rightarrow \bar{B}_1^E$ is weak seq. cpt. (equi. in fact.)

Pf: For $(x_n : \|x_n\| \leq 1) \subset E$. let:

$u = \overline{\text{span}(x_k)}$. sep. $\cong u^{**}$.

since u is also reflexive.

$\Rightarrow u^*$ is sep. $\Rightarrow (J_u(x_k))$ has weak* - conv. subseq. $\to X^{**}$.

So: $x_{n_k} \xrightarrow{*} \beta_n^{-1} x^{**} \Rightarrow x_{n_k} \to \beta_n^{-1} x^{**}$.

⑧ Dual operators:

$T \in \mathcal{L}(E, F)$. $T^* \in \mathcal{L}(F^*, E^*)$ is its dual operator defined by $T^* f = f T$. $\forall f \in F^*$.

Rmk: check $\|T\| = \|T^*\|$.

e.g. i) $E = F = \ell_p$. $T: (x_1, x_2, \cdots) \in \ell_p \mapsto (x_2, x_3 \cdots) \in \ell_p$

$T^* y(x) = y T(x) = \sum x_{k+1} y_k$. $y \in \ell_q$.

$= z(x)$. So: $T^* y = z = (0, y_1, y_2, \cdots) \in \ell_q$.

i.e. $T^*: (y_1, y_2, \cdots) \mapsto (0, y_1, \cdots y_n \cdots)$ and

$\|T^*\| = \|T\| = 1$.

ii) $F \subset E$. n.v.s. $i: F \hookrightarrow E$ inclusion map

$i^* y(x) = y i x = y(x)$. $\forall y \in E^*$.

So: $i^* y = y|_F$ restriction operator.

Rmk: Given $\|\cdot\|_F, \|\cdot\|_E$. If $i \in \mathcal{L}(F, E)$

$\Rightarrow i^* \in \mathcal{L}(E^*, F^*)$. denote $j = i^*$.

In fact. $E^* \subset F^*$ means:

$j: f \in E^* \mapsto f|_F \in F^*$. rather set inclusion relation. (i.e. $E^* \cong Z_1$. $F^* \cong Z_2$. $Z_1, Z_2 \subset Z$. $Z_1 \subset Z_2$) e.g.

$E = l_1$. $E^* \cong l_\infty$. $F = \mathrm{span}\{e^k\}_1^n$. $F^* \cong F$.

But $l_\infty \not\subseteq \mathrm{span}\{e^k\}_1^n$.

Lem. $T \in L(E, F)$. Then: T^* is injective $\Leftrightarrow T(E) \subset F$ is dense.

Pf: ($\Rightarrow$) By Hahn-Banach. $\exists\, l \in F^*$. s.t. $l|_{\overline{T(E)}} = 0$. $l(y_0) \neq 0$. $y_0 \in F / \overline{T(E)}$

$\Rightarrow T^*(l) = 0$. but $l \neq 0 \in \ker(T^*)$.

($\Leftarrow$) $\forall\, l \in \ker(T^*)$. $0 = T^* l = l \circ T$

So: $l = 0$ since $T(E)$ dense in F.

Lem. E, F Banach. $T \in \mathcal{L}(E, F)$. Then:

i) $N(T^*) = R(T)^\perp$ ii) $N(T) = R(T^*)_\perp$

iii) $N(T)^\perp = \overline{R(T^*)}$ iv) $N(T^*)_\perp = \overline{R(T)}$.

v) $N(T^*)$ is weak* seq closed in F^*.

vi) $R(T)$ closed $\Leftrightarrow R(T^*)$ closed.

Pf: Check i) - v) directly. (By $\overline{U} = U^{\perp\perp}$)

vi) Only consider $E = H_1$, $F = H_2$ Hilbert.

For $(\Rightarrow)$. Set: $\hat{T} : N(T)^{\perp} \to R(T)$

$x \mapsto Tx$

$\Rightarrow$ $\hat{T}$ is BLO. bijective. $R(T)$ Hilbert.

So: $\hat{T}^{-1}$ is BLO by open mapping Thm.

$\varphi : Tx \in R(T) \mapsto \langle \hat{T}^{-1} Tx, y \rangle \in \mathbb{K}$. for some fix $y \in \overline{R(T^*)}$.

$\Rightarrow \varphi \in \mathcal{L}(R(T))$. $\exists z \in R(T)$. s.t.

$\langle \hat{T}^{-1} Tx, y \rangle = \langle Tx, z \rangle = \langle x, T^* z \rangle$

let $x = x_1 + x_2 = N(T) \oplus N(T)^{\perp}$.

$\Rightarrow \langle x_2, y \rangle = \langle x_2, T^* z \rangle$. So $y \in R(T^*)$.

$y \in N(T)^{\perp}$