Hilbert Space

e.g. (inner product).

$W \in C([0,1], \mathbb{R}^1)$. $\langle f, g\rangle_W = \int_0^1 f \bar{g} W dt : C[0,1]^{\otimes 2} \to \mathbb{C}^1$. $\Rightarrow$ $\langle f, g\rangle_W = \overline{\langle g, f\rangle_W}$. and

$\langle \lambda f_1 + f_2, g\rangle_W = \lambda_1 \langle f_1, g\rangle_W + \langle f_2, g\rangle_W$

Claim: $\langle\ ,\ \rangle_W$ is inner product $\Leftrightarrow$ $\forall x \in [0,1]$ $\forall I_x \ni x$. $|I_x| > 0$. $\exists x_0 \in I_x$. s.t. $W(x_0) > 0$. and $W(x) \geq 0$. (proved by contradiction.)

Def: ONS collects a family of pairwise orthogonal elements. s.t. $\|u_n\| = 1$.

ONB B is complete/maximal if B is ONS and $\forall \tilde{B}$ ONS. $\tilde{B} \supseteq B \Rightarrow \tilde{B} = B$.

Rmk: i) By Zorn's Lem. ONB exists.

ii) ONS B is complete $\Leftrightarrow$ $B^{\perp} = \{0\}$.

Lem. H is inner product space. (u_n) is ONS.

i) (Bessel inequi.) $\sum |\langle x, u_n\rangle|^2 \leq \|x\|^2$.

Besides, if H is Hilbert space, then:

$\sum_1^\infty \langle X, u_j \rangle u_j \to \tilde{X}$. (may not be X).

ii) If H is Hilbert space, then:

(u_n) is ONB $\overset{a)}{\Longleftrightarrow}$ $\forall x \in H$. $X \overset{uni.}{=} \sum_1^\infty \langle X, u_j \rangle u_j$

$\overset{b)}{\Longleftrightarrow}$ (Parseval id.) $\forall x \in H$. $\|x\|^2 = \sum |\langle X, u_j \rangle|^2$

Pf: i) By $0 \le \|X - \sum_1^n \langle X, u_j \rangle u_j\|^2 = \|X\|^2 - \sum_1^n \square$

ii) $\vec{a)}$ Check $\langle X - \sum \langle X, u_j \rangle u_j, u_m \rangle = 0$ $\forall m$.

$\vec{b)}$ By conti. of $\langle \cdot, \cdot \rangle$.

$\overleftarrow{b)}$ $\langle X, u_j \rangle = 0$. $\forall j \Rightarrow X = 0$.

Thm. H is ∞-dim Hilbert space. Then:

i) H is sep. ii) H has countable ONB

iii) $H \cong \ell_2$. We have: i) $\Longleftrightarrow$ ii) $\Longleftrightarrow$ iii).

Pf: i) $\Rightarrow$ ii). (x_n) is dense countable in H.

$\forall u \in B$, $\exists x_u$. st. $\|x_u - u\| \le \frac{1}{2}$.

So: $\sqrt{2} = \|u - v\| < 1 + \|x_u - x_v\|$. $\forall u, v \in B$.

ii) $\Rightarrow$ iii) $T: X \in H \mapsto (\langle X, u_n \rangle) \in \ell_2$.

iii) $\Rightarrow$ i) is trivial.

e.g. ONB of $L^2(\mathbb{R}^1)$: Hermite functions

$$h_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Non-Sep. Hilbert:

Def: $(x_i)_I \subseteq E$. n.v.s. Then: $\sum_I x_i$ converges unconditionally to $x \in E$. written $x = \sum_I x_i$.

if a) $J = \{ j \in I \mid x_j \neq 0 \}$ is countable.

b) If $|J| < \infty$. $\Rightarrow x = \sum_J x_j$.

If $|J| = \infty$. $\Rightarrow x = \sum_k x_{j_k}$. $\forall$ order (j_k).

Rmk: Alternatively definition is a) +

b') $\sum_1^\infty \varepsilon_j x_j$ converges for $\forall \varepsilon_j \in \{\pm 1\}$.

Thm. For E is Banach.

i) Absolutely conv. $\Rightarrow$ unconditionally conv.

ii) If $\dim E = \infty$. Then $\exists (x_i)_I$ conv. uncond. but not absolutely.

Rmk: $\dim E < \infty$. Then they're equi.

Pf: i) $\sum \|x_j\| < \infty$. So: $\#\{ x_j \mid |x_j| \neq 0 \} \leq \aleph_0$.

And $\| \sum \varepsilon_j x_j \| \leq \sum \|x_j\| < \infty$.

ii) let $E = \ell_p$. $1 < p \le \infty$. $\sum_1^\infty \frac{1}{n} e^n$. is not abs. conv. : $\sum \|\frac{1}{n} e^n\|_{\ell_p} = \sum \frac{1}{n} = \infty$.

But $\forall$ permutation τ.

$1 < p < \infty$: $\|\sum \frac{1}{\tau(n)} e^{\tau(n)}\|_p^p = \sum \tau(n)^{-p} < \infty$

$p = \infty$: $\|\sum \frac{1}{\tau(n)} e^{\tau(n)}\|_\infty = \sup \tau(n)^{-1} < \infty$.

<u>Dual space :</u>

Riesz Thm : For $f \in H^*$. $\exists z'$. $f(z') = 1$. Denote $z' = z + \tilde{z} \in \ker(f)^\perp \oplus \ker(f)$. We have $f = \gamma_{z/\|z\|^2}$.

<u>Rmk</u>: Different with n.v.s. case. The extension of BLO on $L \subset H$ will be unique. (H is uniformly convex)

For $T \in \mathcal{L}(H_1, H_2)$. its adjoint operator $T^* \in \mathcal{L}(H_2, H_1)$ defined by $\langle Tx, y \rangle = \langle x, T^* y \rangle$.

<u>Rmk</u>:

$$\begin{tikzcd} H_1 \arrow[r, shift left, "T"] & H_2 \arrow[l, shift left, "T^*"] \arrow[d, "l_2"] \\ H_1^* \arrow[u, "l_1^{-1}"] & H_2^* \arrow[l, "\tilde{T}^*"] \end{tikzcd}$$

Assume l_1, l_2 are Riesz iso. relation of dual op. $\tilde{T}^*$ and adjoint op. T^* is :

$$T^* = l_1^{-1} \circ \tilde{T}^* \circ l_2.$$

Self-Adjoint / cpt Operator:

Lem. H is inner product space. $T \in L(H)$ is self-adjoint. $\Rightarrow \|T\| = \sup_{\|x\|\le 1} |\langle Tx, x\rangle|$.

Pf: Set $c = RHS$. $\forall x, y \in H$.

$\langle T(x+y), x+y\rangle \le c\|x+y\|^2$.

$\langle T(y-x), y-x\rangle \le c\|x-y\|^2$.

$\Rightarrow 4\mathrm{Re}\langle Tx, y\rangle \le c(\|x+y\|^2 + \|x-y\|^2)$

$= 2c(\|x\|^2 + \|y\|^2)$

from addition of two inequi. and

$\langle Ty, x\rangle + \langle Tx, y\rangle = 2\mathrm{Re}\langle Tx, y\rangle$ by $T = T^*$.

Let $y = Tx\|x\|/\|Tx\| \Rightarrow \|T\| \le c$

Lem. H is Hilbert. (u_n) is ONS. $(\lambda_i) \to 0$.

$\Rightarrow Tx = \sum_1^\infty \lambda_i \langle x, u_i\rangle u_i \in K(H)$.

Cor. $T \in K(\ell_2)$ for $T(x) := (\lambda_n x_n)$ st. $\lambda_n \to 0$.

Pf: $T_n := \sum_1^n \lambda_k \langle x, u_k\rangle u_k$.

$\Rightarrow \|T - T_n\| \le \max_{j \ge n} |\lambda_j| \to 0$.