Linear FPEs

(1) Background:

Consider $dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$.

$b: \mathbb{R}^{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d$. $\sigma: \mathbb{R}^{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^{d \times k}$.

Denote $A = (a_{ij}) = \frac{1}{2}\sigma\sigma^T$. $\mu_t \triangleq \mathcal{L}(X_t)$

By Itô's formula, we have:

$$\int_{\mathbb{R}^d} \varphi(x) d\mu_t = \int_{\mathbb{R}^d} \varphi \, d\mu_0 + \int_0^t \int_{\mathbb{R}^d} a_{ij}(s,x) \partial_{ij}^2 \varphi(x)$$

$$+ \int_0^t \int_{\mathbb{R}^d} b_i(s,x) \partial_i \varphi(x) d\mu_s(x) ds. \quad \forall \varphi \in C_c.$$

$\overset{\partial t}{\Rightarrow}$ Its distributional formulation is:

$\partial_t \mu_t = \partial_{ij}^2 (a_{ij}\mu_t) - \partial_i (b_i \mu_t)$ (Einstein Sum)

<u>Remark</u>: i) $\mathcal{M}_b^+(X)$ is set of nonnegative finite Borel measures on X.

metric space; $\mathcal{M}_b^+ \triangleq \mathcal{M}_b^+(\mathbb{R}^d)$.

ii) $\mathcal{P}(X)$ is set of s.p.m's on X.

Recall: i) In polish space. Weak converges

(for $\forall C_B$) $\Leftrightarrow$ Vague converges

(for $\forall C_c$) + tightness.

If X is only locally cpt.

$\Rightarrow$ Only Vague conv is weak*.

e.g.: $(\delta_n) \xrightarrow{v} 0$. But not weakly

ii) If X is cpt Hausdorff. Then

weak, vague conv. are both

weak* conv. (Since $C_0 = C_c = C_B$)

(2) Definitions:

Set $d \in \mathbb{N}$. $b = (b_i)_1^d$. $a = (a_{ij})_{d \times d}$. where.

$b_i, a_{ij}, c : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^1$. $a \geq 0$. Sym. $\forall t, x$.

We want to study:

$\partial_t \mu = \partial_{ij}^2 (a_{ij}(t,x)\mu) - \partial_i (b_i(t,x)\mu) + c(t,x)\mu$

Set $L_{a,b,c} : \varphi \mapsto L_{a,b,c}\varphi = a_{ij}(t,x)\partial_{ij}^2 \varphi(x) + b_i(t,x)$

$\partial_i \varphi(x) + c(t,x)\varphi(x)$. And Denote L^* is dual

$\Rightarrow \langle L\varphi, \mu \rangle = \langle \varphi, L^*\mu \rangle$.

We can write the PDE in $\partial_t \mu = L^*_{a,b,c}\,\mu$

Def: i) A locally finite Borel measure on

$(0,\infty) \times \mathbb{R}^d$ satisfies FPE if for

$a_{ij}, b_i, c \in L^1_{loc}((0,\infty)\times\mathbb{R}^d, \mu)$. And

$\forall \varphi \in C_c^\infty(\mathbb{R}^{>0} \times \mathbb{R}^d)$. We have:

$\int_{\mathbb{R}^{>0}\times\mathbb{R}^d} \partial_t \varphi + L_{a,b,c}\,\varphi \, d\mu(t,x)$.

Rmk: We restrict on: $\mu(t,x)$ is Borel curve. i.e. $\mu_t(\cdot)$ is Borel. $\forall t \geq 0$ and it's locally finite, nonnegative with form $d\mu(t,x) = d\mu_t(x)dt$ (But not every LF Borel $\mu \geq 0$ has this form. Require disintegration measure ν of $\mu_t(x)$ w.r.t t $\ll dt$)

ii) μ is solution with const. mass if $\forall t,s > 0$. $\mu_t(\mathbb{R}^d) = \mu_s(\mathbb{R}^d)$.

iii) $(\mu_t)_{t\geq 0}$ is FPE solution with initial value $\nu \in \mathcal{M}_B^+$. if $\forall \varphi \in C_c^\infty(\mathbb{R}^d)$.

$\exists\, O_\varphi \subseteq (0,\infty)$. full dt-measure set s.t. $\int_{\mathbb{R}^d} \varphi \, d\nu = \lim_{t\in O_\varphi \to 0} \int_{\mathbb{R}^d} \varphi \, d\mu_t$.

Rmk: i) We call FPE + this initial cond. by Cauchy problem

ii) $\mu_t \xrightarrow{v} \nu \ (t\to 0) \Rightarrow O_\varphi^c = \varnothing$. $\forall \varphi \in C_c^\infty$

iii) The initial value ν is unique

iv) Locally finite Borel measure $(\mu_t)_{t\geq 0}$ solve FPE with datum ν. if a_{ij}, b_i, $c \in L^1_{loc}(\mathbb{R}^{\geq 0}\times\mathbb{R}^d, dx \otimes dt)$. and $\forall \varphi \in C_c^\infty$. $\exists$ dt-full measure set $J_\varphi \subset (0,\infty)$ s.t. $\int_{\mathbb{R}^d} \varphi \, d\mu_t = \int_{\mathbb{R}^d} \varphi \, d\nu + \lim_{\tau\to 0^+} \int_\tau^t \int_{\mathbb{R}^d} L_{a,b,c}\varphi \, d\mu_s ds \quad \forall t \in J_\varphi$. $(*)$.

Rmk: i) (μ_t) may not be unique. since J_φ depends on φ. Approxi. of φ can't work.

ii) Let $t\to 0$. Note O_φ can be J_φ !

Lemma*. For $(\mu_t)_{t\geq 0}$ solves FPE with ν

i) If $a_{ij}, b_i, c \in L^1_{loc}([0,\infty)\times\mathbb{R}^d, \mu_t dt)$. Then for $\forall \varphi \in C_c^\infty$. $\lim_{\varepsilon\to 0^+} \int_\varepsilon^t \int_{\mathbb{R}^d} L_{a,b,c}\varphi \, d\mu_s ds = \int_0^t \int_{\mathbb{R}^d} L_{a,b,c}\varphi \, d\mu_s ds$.

Besides, in this case. $J_\varphi^c = \emptyset$. $\Leftrightarrow$ $t \mapsto \mu_t$ is vague conti. on $[0,\infty)$

ii) $\forall T$, $a_{ij}, b_i, c \in L^1([0,T]\times\mathbb{R}^d, \mu_t dt)$, $t\mapsto \mu_t$ is vague conti. on $[0,\infty)$ for $\forall T$. Then: $(\mu_t)_{t\geq 0}$ satisfies

$\mu_t(\mathbb{R}^d) = \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(s,x)\, d\mu_s(x) ds$

for $\forall t \geq 0$.

And in this case. $(*)$ holds for $\forall \varphi \in C_b^2(\mathbb{R}^d)$.

Pf: i) First note $t \mapsto \int_{\mathbb{R}^d} L_{a,b,c}\varphi \, d\mu_t \in L^1_{loc}([0,\infty))$. (We take $[0,\infty)$ rather $(0,\infty)$ here!) And so $t\mapsto \int_0^t \square_{(s)}$ conti. $\Rightarrow$ $t\mapsto \mu_t$ is v-conti.

ii) Note $J_\varphi = 0$ in this case. So we can approxi. $\forall \varphi \in C_{ub}$. and set $\varphi = 1$. combine with i)

<u>Rmk</u>: For $c=0$. (μ_t) has const. mass as $\gamma(\varphi^\lambda)$.

<u>prop.</u> $(\mu_t)_{t\geq 0}$ given by $d\mu(t,x) = d\mu_t(x)dt$ satisfies Ref i) and iii) $\Longleftrightarrow$ it satisfies Ref iv).

(b) <u>Existence</u>:

<u>Steps</u>: 1°) Approxi. coefficients and initial datum by regular one where the existence is easy to get.

2°) Prove uniform estimate for the solutions in 1°) to extract a convergent subseq.

3°) The limit solve original one.

Lemma[1] (Existence)

For $0 < m < M$. fixed const. s.t.

$mI \leq a(t,x) \leq MI$ (in sense of positive definite). If $a_{ij} \in C_B^2$.

$b_i \in C_B^1$, $c \in C_B$. with their derivative are all q-Hölder in x. uniform at t.

And $|a_{ij}(t,x) - a_{ij}(s,y)| \leq C(|x-y|^q + |t-s|^{q/2})$

Then: $\forall$ prob. density $\rho_0 \in C_B(\mathbb{R}^d)$.

$\exists$ s.p.m solution $(\mu_t)_{[0,T)}$ solves FPE

with $\nu = \rho_0 dx$. s.t. $\mu_t = \rho_t dx$.

Besides. $\rho_t(x) \in C^{1,2}((0,T)\times\mathbb{R}^d) \cap C([0,T)\times\mathbb{R}^d)$.

and $\forall$ a.e.-dt $t \in (0,T)$. We have:

$\mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c \, d\mu_s ds$.

Lemma[2] (Estimate).

$(\mu_t)_{t\in(0,T)}$ is solution to FPE. If on $\forall J \times U \subset (0,T)\times\mathbb{R}^d$. a is bdd

Lip-conti. in x uniform at t. $J = [t_0, T-t_0]$

And $\exists\, m(J,U) > 0$. const. s.t.

$m(J,U) I \leq a(t,x)$. $\forall (t,x) \in J \times U$.

Then $\mu_t = \varrho_t(x)dx$, $\varrho \in L^{\frac{d+3}{d+2}}_{\text{loc}}((0,T)\times\mathbb{R}^d)$, *and for every* $J \times U$ *and every neighborhood* W *of* $J \times U$ *with compact closure in* $(0,T)\times\mathbb{R}^d$ *one has*

$$|\varrho|_{L^{\frac{d+3}{d+2}}(J\times U)} \leqslant \bar{C},$$

where $\bar{C}$ *depends on* $d, \inf_W \det a, |a|_{L^\infty(W)}, |b|_{L^1(W;\mu)}, |c|_{L^1(W;\mu)}$, J, U, W *and* $\mu_t dt(W)$.

Prop. (Existence)

If for $c_0 > 0$. $c \leq c_0$. a_{ij}. b_j. c are all bdd on each $(0,T) \times U$. and assume $\exists$ const. $0 < m(U) < M(U)$. s.t.

$m(U) I \leq a(t,x) \leq M(U) I$. on $(0,T) \times U$.

Then: $\forall\, \nu$ p.m. $\exists$ s.p.m solution $(\mu_t)_{[0,T)}$ for FPE with initial ν. s.t. $c \in L^1((0,T)\times\mathbb{R}^d;\, \mu_t dt)$. for a.e-$t$, $t \in (0,T)$.

$\mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c\, d\mu_t dt$.

Pf: 1) WLOG, set $c_0 = 0$. Extend $a_{ij} = \delta_{ij}$, $b_i = c = 0$ on $[0,T)^c \times \mathbb{R}^d$.

Set (w_ε) is mollifier s.t. $w_\varepsilon = 0$ on $B^c_{\varepsilon^{-1}}$, $w_\varepsilon \geq 0$.

$a^n_{ij} := a_{ij} * w_{\frac{1}{n}} + n^{-1}\delta_{ij}$. $b^n_i := b_i * w_{\frac{1}{n}}$ and

$c^n := c * w_{\frac{1}{n}}$. all satisfy the conds

of Lem¹. Note $a^n \geq n^{-1} I$. So it also

satisfies conds of Lem².

2) Set $U_k := B(0,k)$. $\Rightarrow (a^n_{ij}), (b^n_i), (c^n)$

converges in $L^p([0,T) \times U_k)$

Set $\nu_n = \eta_n \, dx \xrightarrow{w} \nu$. $\eta_n \in C^\infty_c$.

For $\partial_t \mu^n = \partial^2_{ij}(a^n_{ij}\mu^n) - \partial_i(b^n_i \mu^n) + c^n \mu^n$.

$\mu^n|_{t=0} = \nu_n$, $\exists$ s.p.m. solution (μ^n_t).

s.t. $\mu^n_t(\mathbb{R}^d) \leq \nu^n(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c^n \, d\mu^n \, ds$

3) We have $m_k = m(U_k)$ const. indept of n.

So, $a^n(t,x) \geq m_k I$. by a is bdd.

And $|a^n|_{L^\infty([0,T) \times U_k)} \leq |a|_{L^\infty([0,T) \times U_{k+1})} + 1$.

Similarly for b^n, c^n.

Next, we consider $\mu^n_t = \varrho^n_t(x) \, dx$ localized on $[T/k, T(1-1/k)] \times W_{k-1} \triangleq V_k$. $k \geq 2$.

$$\int_{V_k} \ell_t^n(x)^{\frac{d+3}{d+2}} \, dx \, dt \leq C_k \quad \text{(indept of } n\text{)}$$

By reflexive of $L^{\frac{d+3}{d+2}}$. $\exists (\ell^{n_k})$. st.

$(\ell^{n_k}) \xrightarrow{w} \ell(t,x)$. Set $\mu_t = \ell_t(x)dx$.

4°) For $\varphi \in C_c^\infty$. $|\int \varphi \, d\mu_t^n - \int \varphi \, d\mu_s^n| =$

$|\int_s^t \int L_{a^n, b^n, c^n} \varphi \, d\mu_r^n dr| \leq C_\varphi |t-s|$

$\Rightarrow \{t \mapsto \int \varphi \, d\mu_t^n\}$ is equiconti. bdd

By Ascoli: $\int_{\mathbb{R}^d} \varphi \, d\mu_t^n(x)$ also admits

a subseq $\xrightarrow{u} \int \varphi \, d\mu_t(x)$ on $(0,T)$.

and $\mu_0^n \xrightarrow{w} \nu$, whose limit coincide

with limits in 3°)

$\Rightarrow \int_{\mathbb{R}^d} \varphi \, d\mu_t^n \xrightarrow{w} \int \varphi \, d\mu_t$ on $[0,T)$.

et-a.s.

5°) Next, we show $(\mu_t)_{[0,T)}$. is Sol.

of FPE with ν.

Note $|\int \varphi \, d\mu_t^n - \int \varphi \, d\nu_n - \int_s^t \int L_{a^n, b^n, c^n} \varphi \, d\mu_r^n dr|$

$\overset{\text{def}}{=} |\int \varphi \, d\mu_s^n - \int \varphi \, d\nu^n| \leq C_\varphi s$. (Δ)

since by def. $L^{n}_{a^n,b^n,c^n}\varphi \xrightarrow{L^p} L_{a,b,c}\varphi$.

with 3'), 4'). first let $n\to\infty$ in (A).

Then set $\delta\to 0$. in (B).

6') Mollifier $(\phi_N)\in C_c^\infty$. $\phi_N=1$ on B_N.

$\phi_N\in[0,1]$. By Lem 2. we have:

$\int \phi_N \, d\hat{\mu}_t^n \le \hat{\mu}_t^n(\mathbb{R}^n)$

$\overset{2)}{\le} \nu_t^n(\mathbb{R}^d) + \int_0^t\int_{\mathbb{R}^d} \phi_N c^n \, d\mu_r^n \, dr$

First set $n\to\infty$. Then set $N\to\infty$

together with Fatou's Lem.

(3) Uniqueness:

Next, assume $a_{ij}, b_i : [0,T]\times\mathbb{R}^d \to \mathbb{R}^1$. both Borel

measurable. $a\ge 0$. symmetric. $c\le c_0=0$.

Prop. If $c=0$. a, b. satisfy $\int_0^T |a(t)|_{C_b^2}+|b(t)|_{C_b^2}<\infty$

Then: FPE has a unique weakly consi-

Solution $(\mu_t)_{(0,T)}$. having const. mass for

$\forall$ initial datum $\nu\in\mathcal{M}_b^+$.

Rmk: $\nu \in \mathcal{P}$ above $\Rightarrow$ $(\mu_t) \subset \mathcal{P}$.

Def: $\mathcal{SP}_\nu := \{ \mu = (\mu_t)_{[0,T]}$ with initial $\nu \in \mathcal{SP}$ (FPE solution)

$| \, c \in L^1([0,T] \times \mathbb{R}^d, \mu_t dt),\ b \in L^2((0,T) \times U, \mu_t dt)$

$\forall U \underset{\text{ball}}{\subseteq} \mathbb{R}^d$. satisfy $\mu_t(\mathbb{R}^d) \le \nu(\mathbb{R}^d) + \int_0^t c \, d\mu_s ds \}$ (a.e.-a.s.)

Assumption: $\forall$ ball $U \subseteq \mathbb{R}^d$.

i) $\exists m(U), M(U) > 0$. s.t. $\begin{cases} a(t,x) \ge m(U) I_d \\ |a(t,x)| \le M(U) I_d \end{cases}$

$\forall (t,x) \in (0,T) \times U$.

ii) $\exists \Lambda(U) > 0$. s.t. $\forall i,j$. $\forall x,y \in U$. $\forall t \in (0,T)$

$: |a_{ij}(t,x) - a_{ij}(t,y)| \le \Lambda(U)|x-y|$.

prop[1]: If Assumpt i). ii) holds. $b \in L^2((0,T) \times \mathbb{R}^d)$ and $c \in L^{\frac{p}{2}}((0,T) \times \mathbb{R}^d))$. for some $p > d+2$. if $\exists (\mu_t)_{t \in [0,T]} \in \mathcal{SP}_\nu$. s.t. $\dfrac{|a_{ij}|}{1+|x|^2} + \dfrac{|b_i|}{1+|x|}$ $\in L^1((0,T) \times \mathbb{R}^d; \mu_t dt)$. Then: we have: μ is unique in $\mathcal{SP}_\nu$.

prop[2]: If assumpt i). ii). hold. $b \in L^p((0,T) \times \mathbb{R}^d)$. & $c \in L^{p/2}((0,T) \times \mathbb{R}^d)$ for some $p > d+2$. Besides

$\exists\, V \in C^2(\mathbb{R}^d; \mathbb{R}_{\geq 0})$. s.t. $V \xrightarrow{|x|\to\infty} \infty$. and

$L_{a,b,c} V(t,x) \leq C + C V(x)$. $\forall (t,x) \in {(0,T)\times \atop \mathbb{R}^d}$

for some $C>0$. Then : $\# SP_0 = 1$.

<u>Rmk</u>: i) V is called lyapunov function.

ii) In prop' & prop². the unique

solution also satisfy :

$\mu_t(\mathbb{R}^d) = V(\mathbb{R}^d) + \int_0^T \int_{\mathbb{R}^d} c \, d\mu(dx)$.

So $c=0$. V is p.m. $\Rightarrow (\mu_t)$ is p.m.

<u>prop</u>. $b \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $a = id_{d\times d}$. $c=0$. $\Rightarrow$ The

solutions of FPE has more than one

<u>Rmk</u>: Good regularity doesn't mean it's

good enough to have unique sol.