Superposition Principle

Next, we suppose $c \equiv 0$. $\Rightarrow L_{a,b}\varphi = a_{ij}\partial_{ij}^2\varphi + b_i\partial_i\varphi$

Equip $C_t\mathbb{R}^d := C(\mathbb{R}_+;\mathbb{R}^d)$ with topo of locally uniform convergence. And set $z_t w = w(t)$.

(1) Mart Problem:

Def: Borel p.m. $\mathbb{P} \in \mathcal{P}(C_t\mathbb{R}^d)$ solves mart. prob. with (a,b) if $\int_{C_t\mathbb{R}^d}\int_0^T |a_{ij}(s,z_s)| + |b_i(s,z_s)| \, ds \, d\mathbb{P} < \infty$ and $\forall \varphi \in C_B^2(\mathbb{R}^d)$. $M_t^\varphi := \varphi \circ z_t - \int_0^t L_{a,b}\varphi(s,z_s)\, ds$ is $\mathbb{P}$-mart. w.r.t $\mathcal{F}_t = \sigma(z_s, 0 \le s \le t)$

Denote: $MP_\nu(a,b)$ is set of solutions above with initial and. $\mathbb{P} \circ z_0^{-1} = \nu$.

Rmk: If start at time $= s$. and consider solution are p.m. on $C([s,\infty), \mathbb{R}^d)$.

then we denote set of s.l. by $MP_{s,\nu}(a,b)$. And $\pi^s_t : C_s \to C_t$, proj.

<u>Lemma.</u> i) $\nu \in \mathcal{P}$. $s \geq 0$, $\mathbb{P} \in MP_{s,\nu}(a,b)$. Set: $(Q_x)_{x \in \mathbb{R}^d} \subseteq \mathcal{P}(C([s,\infty),\mathbb{R}^d))$ be ν-a.s. unique disintegration family s.t. $x \mapsto Q_x(A)$ is measurable. $\forall A \in \mathcal{B}_{C_s,\mathbb{R}^d}$ and $\mathbb{P}(A) = \int_{\mathbb{R}^d} Q_x(A)\, d\nu(x)$.

$\Rightarrow Q_x \in MP_{s,x}(a,b)$. for ν-a.s. x.

<u>Rmk:</u> Converse is true.

ii) $t_i \geq s$. $Y = (\pi^s_{t_0}, \pi^s_{t_1} \cdots \pi^s_{t_n}) : C([s,\infty),\mathbb{R}^d) \to (\mathbb{R}^d)^{n+1}$. $\mathcal{A} = \sigma(Y)$. Set $\mathbb{P} \in MP_{s,\nu}(a,b)$ and $Q_w(\cdot)$ is r.c.p. of $\mathbb{P}$. w.r.t. $\mathcal{A}$. Denote $Q^{t_n}_w(\cdot)$ is restriction of Q_w on $\mathcal{B}_{C([t_n,\infty),\mathbb{R}^d)}$

i.e. $Q^{t_n}_w(A) = Q_w(\{u \in C([s,\infty),\mathbb{R}^d) \; u|_{[t_n,\infty)} \in A\})$ $\forall A \in \mathcal{B}_{C([t_n,\infty),\mathbb{R}^d)}$.

Thm: $\exists A \in \mathcal{A}$. s.t. $\mathbb{P}(A) = 0$ and

$Q_w^{t_n}(\cdot) \in MP_{t_n, w(t_n)}(a, b)$. $\forall w \in A^c$.

Rmk: i) If $\mathbb{P}^1, \mathbb{P}^2 \in MP_{s,v}(a,b)$. s.t.

$\mathbb{P}^1 = \mathbb{P}^2$ on $\mathcal{A}$, $\Rightarrow A$ can be

chosen: $\mathbb{P}^2(A) = \mathbb{P}^1(A) = 0$

ii) It shift initial value to t_n

Prop. X is the weak solution to the SDE

$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$. $\Leftrightarrow$

$\mathcal{L}_X \in MP_v(\frac{1}{2}\sigma\sigma^T, b)$. $v = \mathcal{L}(X_0)$.

Cor. $\mathbb{P} \in MP_v(a, b) \Rightarrow (\mathbb{P} \circ Z_t^{-1})_{t \geq 0}$ is

weakly conti. p.m. solution to

FPE with initial dist. v. and

satisfies Lem$^{(*)}$ in E & U (2)

Pf: Note the SDE has t-conti.

solution satisfies FPE.

(2) Superposition principle:

① Thm (Superposition Principle)

$\sigma_{ij}, b_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^1$. Borel-measurable

Then: $\forall$ weakly conti. p.m sol. $(\mu_t)_{t\geq 0}$

to FPE with $a = (a_{ij})_{i,j\leq d} = \frac{1}{2}\sigma\sigma^T$

and b. st.

$\{\int_0^T \int_{\mathbb{R}^d} |a_{ij}| + |b_i| \, d\mu_t \, dt < \infty, \ \forall T > 0, \ i,j \leq d\}$

$\exists$ weak sol. X to SDE. st. $\mathcal{L}_{X_t} = \mu_t, \ \forall t \geq 0$

Rmk: i) Converse part is from Itô's.

ii) There's no regularity cond. on a, b except measurability.

iii) More generally. $(\cdots)$ can be replaced by $\int_0^T \int_{\mathbb{R}^d} \frac{|a_{ij}| + |\langle b, x\rangle|}{1+|x|^2} \, d\mu_t \, dt < \infty$.

~~But~~ for merely local integrable a, b. it doesn't hold.

iv) Restrict on the p.m's. is necessary

<u>prop.</u> (μ_t) solves $\partial_t \mu_t = L^*_{a,b}\mu_t$ with the initial cond. $\nu \in \mathcal{P}$. s.t. $a_{ij}, b_j \in L^1_{loc}([0,\infty)\times\mathbb{R}^d, \mu_t dt)$ and $\mu_t \in \mathcal{M}_b^+$ and $\operatorname{ess\,sup}_{t\geq 0} \mu_t(\mathbb{R}^d) < \infty$

Then: $\exists$ unique vaguely conti. dt-version $(\tilde{\mu}_t)_{t\geq 0}$ solves FPE with initial ν.

if in addition, $a_{ij}, b_i \in L^1([0,T]\times\mathbb{R}^d, \mu_t dt)$ $\forall T>0$. then $\tilde{\mu}$ is weakly conti. ($\Rightarrow$ p.m.)

<u>Pf:</u> Uniqueness is obvious by $\mu_t = \tilde{\mu}_t$ a.s.

Since $\int_{\mathbb{R}^d} \varphi \, d\mu_{s,t} \overset{\text{loc}}{=} \int_s^t \int_{\mathbb{R}^d} L_{a,b}\varphi \, d\mu_r dr$.

Fix $\varphi \in C_c^\infty$. $\forall s<t \in A_\varphi$. $\mathbb{P}(A_\varphi^c)=0$.

RHS is conti. So for each φ, we have conti modification. Since $L^2(\mathbb{R}^d)$ is separa.

$\exists (\varphi_n) \subset C_c^\infty(\mathbb{R}^d)$. dense. We can have a linear. bdd modifi. on (φ_n). $\forall s,t \in \mathbb{R}^1$. by approxi of $t \in \bigcup A_{\varphi_n}$. $\Rightarrow$ conti. extend on $C_c^\infty(\mathbb{R}^d)$. & apply Riesz repre. Since $\|\mu\|_\infty < \infty$. So $\exists \nu_t$. radon measure. st it's truly $\varphi \mapsto \int \varphi \, d\nu_{s,t}$. a.s.

Cor. For $s \geq 0$, $\nu \in \mathcal{P}$. If solution to the SDE with initial (s, ν) is weakly unique. Then $\exists$ unique sol. to FPE with initial (s, ν). st.

$\int_0^T \int_{\mathbb{R}^d} |a_{ij}| + |b_i| \, d\mu_t \, dt < \infty$ holds. $\forall T > 0$

Pf: WLOG. $s = 0$. if (μ_t^i) p.m.'s solve FPE with ν satisfy the conditions. $i = 1, 2$.

By prop. above. $\exists$ weakly conti. t-version $(\tilde{\mu}_t^i)$ $i = 1, 2$

Apply superposition on $(\tilde{\mu}_t^i)$.

$\exists \tilde{X}_t^i$. solves SDE. st. $\mathcal{L}_{\tilde{X}_t^i} \sim \tilde{\mu}_t^i$

By cond. $\Rightarrow \mathcal{L}_{\tilde{X}_t^1} = \mathcal{L}_{\tilde{X}_t^2}$.

Rmk: Reverse is false. We need more cond. (i.e. know $\forall$ datum)

Prop.(*) If weakly conti. p.m. solutions $(\mu_t)_{t \geq s}$ satisfies $\int_s^T \int_{\mathbb{R}^d} |a_{ij}| + |b_i| \, d\mu_s \, ds < \infty$ are unique for $\forall$ initial $(s, \delta_x) \in \mathbb{R}_+ \times \mathcal{P}$.

Thm: Solutions for the SDE are unique for $\forall (s, \delta_x)$, $\forall s \geq 0$, $\forall x \in \mathbb{R}^d$.

<u>Rmk</u>: Mart. Prob. $\Longleftrightarrow$ Weak sol. to SDE

$$SDE \underset{\text{exist}^{ii)}}{\overset{\text{exist}^{i)}}{\rightleftarrows}} FPE$$

i): Itô formula

ii) superposition prin.

$$SDE \underset{\text{unique}^{(*)}}{\overset{\text{unique}^{iii)}}{\rightleftarrows}} FPE.$$

iii) Cor. above

(*) Need discuss on soft many later

<u>Pf</u>: Fix $x \in \mathbb{R}^d$. $s=0$. WLOG.

We prove $\mathbb{P}^1, \mathbb{P}^2 \in MP_x(a, b) \Rightarrow \mathbb{P}^1 = \mathbb{P}^2$

i.e. $\forall n$, $(t_i)_0^n \subset \mathbb{R}^{\geq 0}$, $t_1 < t_2 \cdots < t_n$

$$\mathbb{P}^1 \circ (Z_{t_0}, \cdots Z_{t_n})^{-1} = \mathbb{P}^2 \circ (Z_{t_0} \cdots Z_{t_n})^{-1}$$

Next, we process by induction:

1) $n=0$. $\mathbb{P}^1 \circ Z_{t_0}^{-1}$, $\mathbb{P}^2 \circ Z_{t_0}^{-1}$ are weakly conti. p.m. solutions satisfies FPE follows from Itô $\Rightarrow \mathbb{P}^1 \circ Z_{t_0}^{-1} = \mathbb{P}^2 \circ Z_{t_0}^{-1}$

2) For $k=n$. Assume $k=n-1$ holds.

prove: $\mathbb{E}_{\mathbb{P}^1}\left(\prod_0^n f_i(Z_{t_i})\right) = \mathbb{E}_{\mathbb{P}^2}\left(\prod_0^n f_i(Z_{t_i})\right)$

Set $(Q^i_w)_{w\in C_+\mathbb{R}^d}$ is r.c.p. of $\mathbb{P}^i$

w.r.t. $\sigma(Z_{t_0}\cdots Z_{t_{n-1}}) \triangleq \mathcal{A}$

Since by induction hypo. $\mathbb{P}^1 = \mathbb{P}^2$ on $\mathcal{A}$.

$\Rightarrow$ [We have $A \in \mathcal{A}$. st. $\mathbb{P}^1(A) = \mathbb{P}^2(A) = 0$.

$Q^{i,t_{n-1}}_w \in MP_{t_{n-1},w_{t_{n-1}}}(a,b)$. $\forall w \in A^c$.]$^{(\Delta)}$

By def of r.c.p. $\exists N_i$. $\mathbb{P}^i(N_i) = 0$.

$$\mathbb{E}_{Q^i_w}(f_n(Z_{t_n})) = \mathbb{E}_{\mathbb{P}^i}(f_n(Z_{t_n}) \mid \mathcal{A}) \quad \forall w \in N_i^c.$$

$\|$

$\mathbb{E}_{Q^{i,t_{n-1}}_w}(f_n(Z^{t_{n-1}}_{t_n}))$. with $n=0$ case:

$$\mathbb{E}_{Q^{1,t_{n-1}}_w}(f_n(Z^{t_{n-1}}_{t_n})) \overset{(\Delta)}{=} \mathbb{E}_{Q^{2,t_{n-1}}_w}(f_n(Z^{t_{n-1}}_{t_n})). \quad \forall w \in A^c.$$

So: $\overset{\|}{H}$ is bdd. $\mathcal{A}$-measurable.

$\mathbb{P}^1$. $\mathbb{P}^2$. a.s. resp. ($\mathbb{P}^i(A^c \cap N_i^c) = 1$. $i=1,2$

But $\mathbb{P}^i(N_1^c \cap N_2^c \cap A^c) \overset{?}{=} 1$)

$$\text{So: } \mathbb{E}_{\mathbb{P}^i}\Big(\prod_0^n f_i(Z_{t_i})\Big) \overset{\text{cond.}}{=} \mathbb{E}_{\mathbb{P}^i}\Big(\prod_0^{n-1} f_i(Z_{t_i})\, H \cdot I_{N_i^c \cap A^c}\Big)$$

which's reduced to $k = n-1$.

Apply induction hypo again.

Prop. $\forall s \geq 0$. If $|MP_{s,x}(a,b)| \leq 1$. $\forall x \in \mathbb{R}^d$.

Then: $|MP_{s,v}(a,b)| \leq 1$. $\forall v \in \mathcal{P}$.

Lemma. (Disintegration Thm)

If i) X, Y are two radon spaces (i.e. $\forall$ Borel p.m is inner regular)

ii) $\mu \in \mathcal{P}(Y)$. $\pi: Y \to X$ is a Borel measurable map. $\left(\begin{matrix}\text{Partition}\\\text{transfer}\end{matrix}\right)$

Thm: Set $\nu = \mu \circ \pi^{-1}$. there exists ν-a.s. uniquely determine family of p.m's $(\mu_x)_x \subseteq \mathcal{P}(Y)$. satisfy:

i) $x \mapsto \mu_x(B)$ is Borel measurable. $\forall B \in \mathcal{B}_Y$.

ii) $\mu_x(\pi^{-1}(\{x\})) = 1$.

iii) $\mu(E) = \int_X \mu_x(E)\, d\nu(x)$.

Pf: By Lemma above: $X = Y = \mathbb{R}^d$. $\mathbb{P} \in MP_{s,v}(a,b)$ uniquely determines ν-a.s. disintegration family (Q_x).

Cor. Under conditions of prop.$^{(*)}$.
it also works for $\forall$ initial
datum (s, ν). $\forall \nu \in \mathcal{P}$.

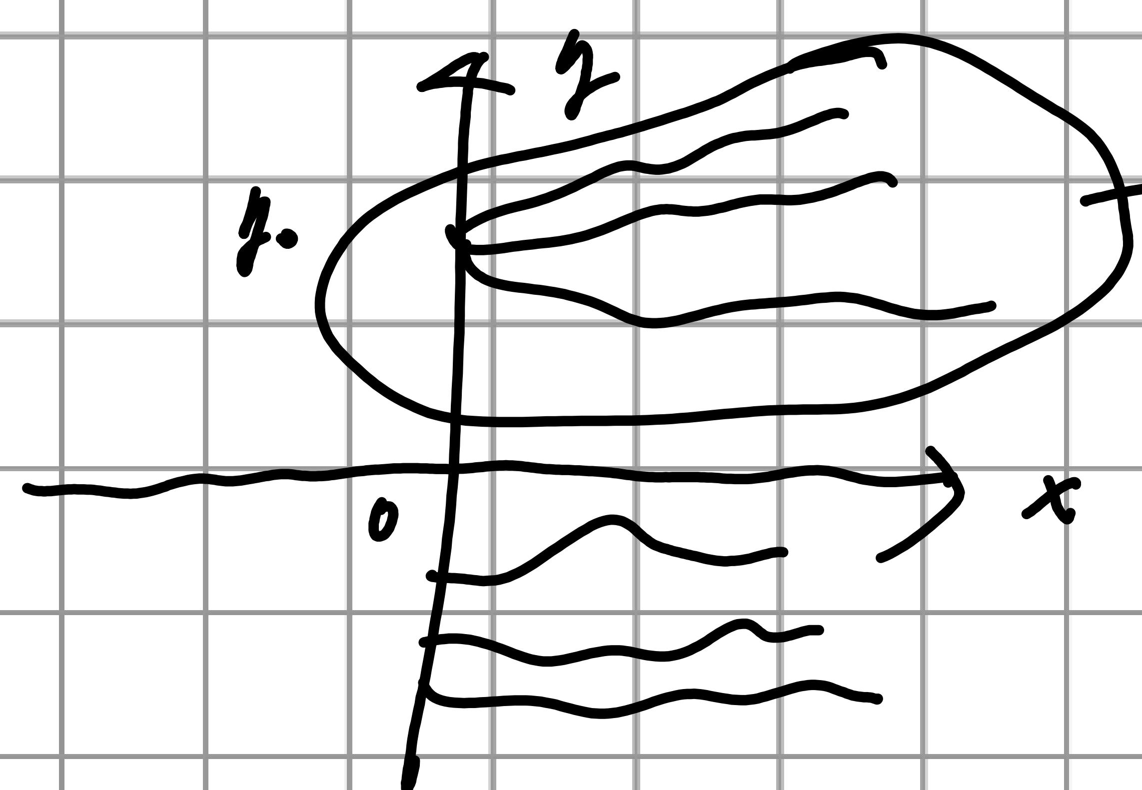
② Deterministic case:

Consider $a=0$. $\partial_t \mu_t = -\operatorname{div}(b\mu_t)$. $t>0$. $(\star)$

Prop. $MP_\nu(0,b) = \{ \mathbb{P} \in \mathcal{P}(C_+\mathbb{R}^d) \mid \mathbb{P}(\eta \in AC(\mathbb{R}^{\geq 0}, \mathbb{R}^d) \mid \eta'(t) \overset{Lt.a.s}{=} b(t, \eta(t))) = 1,\ \mathbb{P}\circ z_0^{-1} = \nu.\ \int_{C_+\mathbb{R}^d}\int_0^T |b(t, w(t))|\,dt\,d\mathbb{P}(w) < \infty.\ \forall T \}$.

Note that Superposition principle asserts:
$\forall$ weakly conti. p.m. solution for $(\star)$. if
$\int_0^T\int_{\mathbb{R}^d} |b|\,d\mu_t\,dt < \infty.\ \forall T>0$. then $\exists \mathbb{P}$. p.m.
supports on set of ODE solutions. s.t.
$\mathbb{P}\circ z_t^{-1} = \mu_t$. by prop. above in the case.
(Exist of weakly conti. $\mu \Rightarrow$ at least one
ODE sol. Conversely, at most one ODE sol.
$\Rightarrow$ uniqueness of weakly conti. μ to FPE)

for disintegration measures $(\mathbb{P}_x)$. $\mathbb{P}_x$ may live on lots of different ODE solutions starting at x. i.e. $\mathbb{P}$ may superpose many ODE sol. with same datum.



$\rightarrow$ $\mathbb{P}_x$ lives on here.

Note $\mathbb{P}_\nu(Z_t \in A) = \int_{\mathbb{R}^d} \mathbb{P}_x(Z_t \in A)\, d\nu(x)$

$= \int_{\mathbb{R}^d} \mathbb{P}_x(\{\omega \in [\text{ODE sol. } \gamma.\ \gamma(0)=x \ \&$

$Z_t(\omega) \in A\}) \, d\nu(x)$.

$\mathbb{P}_\nu$ also lives on $\bigcup_{x \in \mathbb{R}^d} [\text{ODE sol. } \gamma.\ \gamma(0) = x]$.

(3) Proof:

Next, we restrict on $[0,1]$ to prove the superposition principle. The strategy is:

i) Approxi. a, b by more regular a^n, b^n which corresp. FPEs have solution μ^n. St.

$\mu^n \xrightarrow{w} \mu$. $(a^n, b^n) \to (a, b)$. and $\exists \mathbb{P}^n \in MP_{\mu_0^n}(a^n, b^n)$. $\mathbb{P}^n \circ \pi_t^{-1} = \mu_t^n$.

ii) Prove tightness of $(\mathbb{P}^n)$ in $\mathcal{P}(C_{[0,1]}\mathbb{R}^d)$.

iii) $\exists$ subseq $(\mathbb{P}^{n_k}) \xrightarrow{w} \mathbb{P}$. so: $\mathbb{P} \circ \pi_t^{-1} = \mu_t$ and prove $\mathbb{P} \in MP_{\mu_0}(a, b)$.

Lemma: If $\int_0^1 |a(t)|_{C_B^2(\mathbb{R}^d)} + |b(t)|_{C_B^2(\mathbb{R}^d)} dt < \infty$ (A_1)

and $\int_0^1 \int_{\mathbb{R}^d} |a| + |b| \, d\mu_t \, dt < \infty$. $\forall T > 0$

Then: $\exists X$. weakly solve SDE s.t. $\mathcal{L}X_t = \mu_t$.

Pf: Under (A_1). By Picard-Lindelöf Thm

$\Rightarrow$ the SDE has unique weak solution for $\forall$ initial $\nu \in \mathcal{P}$.

π_t's weakly conti. & solve FPE

Note that $\forall$ FPE with datum ν can have only one weakly conti. sol.

So the one-time marginal to the SDE is sol. to the FPE.

Next. we generalize the cond. (A1) gradually

: (A2) $\int_0^1 \sup_x |a(t,x)| + \sup_x |b(t,x)| \, dt < \infty$.

$\Downarrow$

(A3) $\int_0^1 |a(t)|_{L^\infty(\mathbb{R}^n)} + |b(t)|_{L^\infty(\mathbb{R}^n)} \, dt$. $\forall n \in \mathbb{R}^n$.

$\Downarrow$

(A4) $\int_0^1 \int_{\mathbb{R}^n} |a(t,x)| + |b(t,x)| \, d\mu_t \, dt < \infty$.

Rmk: (A4) is case of superposition principle.

Next. we denote: $\mu = (\mu_t)_{[0,1]}$ is the weakly conti. solution of FPE.

① Approxi.

We introduce two methods of approxi.

Rmk: The approxi. should satisfy the regular cond. (A1). which already provides us a easy case that holds.

i) Image of C^2_{ub} - maps:

Set $g = (g^1 \cdots g^d) \in C^2_{ub}(\mathbb{R}^d, \mathbb{R}^d)$. & let

$\mu^g_t := \mu_t \circ g^{-1}$. Note:

$L_{a,b}(\varphi \circ g) = \sum_1^d L_{a,b}(g^k)((\partial_k \varphi) \circ g) + \sum_{\substack{k,\ell \\ i,j}} a_{ij} \partial_i g^k \partial_j g^\ell ((\partial^2_{k\ell} \varphi) \circ g)$

By factorization lemma:

$\exists\, a^g_{k\ell}(t),\ b^g_k(t) : \mathbb{R}^d \to \mathbb{R}^1$. Borel. satisfy

$\mathbb{E}_{\mu_t}(a_{ij}(t) \partial_i g^k \partial_j g^\ell \mid \sigma(g)) = a^g_{k\ell}(t) \circ g$,

$\mathbb{E}_{\mu_t}(L_{a,b}\, g^k(t) \mid \sigma(g)) = b^g_k(t) \circ g$. μ^g_t-a.s. unique.

$\Rightarrow \mu^g$ is weakly conti. solution to $\partial_t V_t = L^{*}_{a^g, b^g} V_t$

$\left(\int_{\mathbb{R}^d} \varphi \, d\mu^g_t = \int \varphi \circ g \, d\mu_t = \int_0^t \int L_{a,b}(\varphi \circ g) = \cdots \right)$

Besides: $\| a^g_{k\ell}(t) \|_{L^p(\mathbb{R}^d, \mu^g_t)} \leq C \| a_t \|_{L^p(\mathbb{R}^d, \mu_t)}$.

$\| b^g_k(t) \|_{L^p(\mathbb{R}^d; \mu^g_t)} \leq C \| |a_t| + |b_t| \|_{L^p(\mathbb{R}^d; \mu_t)}$.

ii) Mollifier:

$\varepsilon : \mathbb{R}^d \to \mathbb{R}^{\geq 0} \in C^\infty$. mollifier. Note that

$L_{a,b}(\varphi * \varepsilon) = \sum_i b_i (\partial_i \varphi) * \varepsilon + \sum_{ij} a_{ij} (\partial^2_{ij} \varphi) * \varepsilon$.

$\mathcal{S}_1$: $\mu_t * \varepsilon$ is weakly conti. solution of

$\partial_t \nu_t = L^*_{a^\varepsilon, b^\varepsilon} \nu_t$. where we define:

$$a_{ij}^\varepsilon(t) = \frac{d((a_{ij}(t)\mu_t)*\varepsilon)}{d(\mu_t*\varepsilon)}(x). \quad b_k^\varepsilon(t,x) = \frac{d((b_i(t)\mu_t)*\varepsilon)}{d(\mu_t*\varepsilon)}(x).$$

which existence is guaranteed by:

<u>Lemma.</u> For $\eta^1 \in \mathcal{M}_B^+$, $\eta^2 \in \mathcal{M}_B$, st. $\eta^2 = h\eta^1$, where

$h: \mathbb{R}^d \to \mathbb{R}^1$. Then:

$d(\eta^2 * \varepsilon) = h_\varepsilon \, d(\eta^1 * \varepsilon)$. i.e. h_ε is its

density. And $\|h_\varepsilon\|_{L^p(\mathbb{R}^d, \eta^1*\varepsilon)} \leq \|h\|_{L^p(\mathbb{R}^d, \eta^1)}$, $\forall p \geq 1$.

② <u>Tightness:</u>

<u>Lemma.</u> S is metric space. $(\mu_n) \in \mathcal{P}(S)$

is tight. $\Longleftrightarrow$ $\exists f: S \to \mathbb{R}_{\geq 0}$. coercive.

i.e. $\{f \leq c\}$ is cpt. $\forall c \geq 0$. and st.

$\sup_n \int_S f \, d\mu_n < \infty$.

<u>prop.</u> θ, Θ_1, Θ_2 : $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. St. Θ_i convex

and $\lim_{x\to\infty} \theta = \lim_{x\to\infty} \Theta_i(x)/x = \infty$.

Thm: $\exists\ \Psi : C_{[0,1]}\mathbb{R}^d \to \bar{\mathbb{R}}_+$, coercive. s.t.

$\mathbb{E}_{\mathbb{P}}(\Psi(f\circ x)) = \int_{\mathbb{R}^d} \theta(|f|)\, d\mathbb{P}_0 + \int_0^1 \Theta_1(|L_{a,b}f|) + \Theta_2(a_{ij}\partial_i f \partial_j f)\, d\mathbb{P}_t\, dt$. $\forall f \in C_b^2(\mathbb{R}^d)$, $\forall \mathbb{P} \in MP_{\mathbb{P}_0}(a,b)$. where $\mathbb{P}_t = \mathbb{P}\circ x_t^{-1}$.

$\Rightarrow$ Note that set $\mathbb{P} = \mathbb{P}_n$. $\forall n$. By choosing appropriate functions θ. Θ_i. $i=1,2$. We have uniform bdd. for $\sup_n \mathbb{E}_{\mathbb{P}_n}(\Psi\circ f\circ x)$.

② <u>Limit</u>:

If $(\mathbb{P}_n)$ is approxi. in ①. having limit to $\mathbb{P}$. Next. We prove: $\mathbb{P} \in MP_{\mathbb{P}_0}(a,b)$.

i.e. $\forall\ \varphi \in C_b^2$. $|\varphi| \leq 1$. $h: C_{[0,1]}\mathbb{R}^d \to \mathbb{R}^1$ conti. bdd. $h \in \mathcal{F}_s$. st.

$\int_{C_{[0,1]}\mathbb{R}^d} h\cdot(\varphi\circ x_t - \varphi\circ x_s - \int_s^t L_{a,b}\varphi(r,x_r)\, dr)\, d\mathbb{P} = 0$

(recall def of cond. expectation)

It holds for $(a^n, b^n, \mathbb{P}^n)$. Substitute the equation above. Note $\mathbb{P}^n \xrightarrow{w} \mathbb{P}$. So:

It's equi to prove:

$$\int_{C([0,1])^k} h\left(\int_s^t L_{a^n, b^n}\varphi(r, Z_r)\,dr\right) dP^n - \int_{C([0,1])^k} h\left(\int_s^t L_{a,b}\varphi(r, Z_r)\,dr\right) dP \to 0 \quad (n\to\infty)$$

i) <u>Using image of p.m.</u>:

Set $g_n \in C_b^2$, s.t. $g_n = x$ on B_n. So that

$D g_n \to I_{d\times d}$, $D^2 g_n \to 0$, $|D^i g_n| \le C$.

Let $\mu^n = \mu^{g_n}$.

Denote $\bar{L} = \bar{a}_{ij}\partial^2_{ij} + \bar{b}_i\partial_i$, where $\bar{a}_{ij}$, $\bar{b}_i$ $\in C_c^\infty$. Then, by Fubini, up to a multiply const. depending on h.

prove: $\overline{\lim_{n\to\infty}} \int_s^t \int_{\mathbb{R}^d} |L_{a^n, b^n}\varphi - \bar{L}\varphi|\,d\mu_r^n\,dr +$

$$\int_s^t \int_{\mathbb{R}^d} |L_{a,b}\varphi - \bar{L}\varphi|\,d\mu_r\,dr \to 0.$$

For the first term:

$$= \int_s^t \int_{\mathbb{R}^d} |\mathbb{E}_{\mu_r}(L_{a,b}(\varphi\circ g_n \mid \sigma(g_n)) - \bar{L}\varphi\circ g_n|\,d\mu_r\,dr$$

contract
$$\leq \int_s^t \int_{\mathbb{R}^d} |L_{a,b}(\varphi \circ g_n) - \bar{L}(\varphi \circ g_n)| \, d\mu_r \, dr$$

$|\nabla \varphi| \leq 1$
$$\leq \int_s^t \int_{\mathbb{R}^d} \sum_{k,l} |a_{ij} \partial_i g_n^k \partial_j g_n^l - \bar{a}_{kl} \circ g_n| +$$
$$\sum_k |L_{a,b}(g_n^k) - \bar{b}_k \circ g_n| \, d\mu_r \, dr$$

$$\xrightarrow{n \to \infty} \int_s^t \int_{\mathbb{R}^d} \sum_{i,j} |a_{ij} - \bar{a}_{ij}| + \sum_k |b_k - \bar{b}_k| \, d\mu_r \, dr$$

For the second term. It also holds:

$$\leq C \int_s^t \int_{\mathbb{R}^d} \sum_{ij} |a_{ij} - \bar{a}_{ij}| + \sum_k |b_k - \bar{b}_k| \, d\mu_r \, dr$$

Since $(\bar{a}_{ij}), (\bar{b}_k)$ are dense in $L^1(\mu_t dt)$

So: $\overline{\lim} \int_s^t \int_{\mathbb{R}^d} |\cdots - \cdots| + \int_s^t \int_{\mathbb{R}^d} |\cdots - \cdots|$ can be arbitrary small.

ii) <u>Using mollifier:</u>

Choose (ℓ_n) smooth. prob. density st. $\ell_n dx \to \delta_0$. Denote $\mu_n := \mu * \ell_n$ and process as above to prove:

$$\overline{\lim_{n \to \infty}} \int_s^t \int_{\mathbb{R}^d} |L_{a^{\varepsilon}, b^{\varepsilon}} \varphi - \bar{L} \varphi| \, d(\mu_r * \ell_n) \, dr$$
$$\leq \int_s^t \int_{\mathbb{R}^d} \sum |a_{ij} - \bar{a}_{ij}| + \sum |b_k - \bar{b}_k| \, d\mu_r \, dr$$

④ Pf of Thm:

1) Under condition (A2). Let $g(x) = Ce^{-\sqrt{1+x^2}}$.

Next, we use mollifier approxi.:

$e_n(x) = n^d g(nx) \to \delta_0$ ($b^i e_n| \leq c n^i e_n$. mollifier

$\Rightarrow$ Set $\mu^n = \mu * e_n \to \mu_t$. $a^n = a^{e_n}$. $b^n = b^{e_n}$.

So: a^n. b^n satisfies (A1) condition. $\exists (\mathbb{P}^n)$

$\in \mathrm{MP}_{\mu_0^n}(a^n, b^n)$. St. $\mathbb{P}^n_t = \mu^n_t$.

2) Since $(\mu_0^n) \to \mu_0$ is tight. $\Rightarrow \exists \theta : \mathbb{R}_+ \to \mathbb{R}^+ \uparrow$.

Coercive. st. $\sup_n \int \theta(|x|) d\mu_0^n \leq 1$.

Apply Vallée Poussin criterion on Cond. (A2).

$\Rightarrow \exists \Theta \uparrow$. convex: $\mathbb{R}^+ \to \mathbb{R}_+$. st. $\lim_{x \to \infty} \frac{\Theta(x)}{x} = \infty$.

and $\int_0^1 \Theta(\sup_x |a|) + \Theta(\sup_x |b|) dt < \infty$.

Denote $X_k : (x_1 \cdots x_n) \mapsto x_k$. coordinate maps

Apply Tightness prop. on θ. $\Theta_1 = \Theta_2 = \Theta$. &

$f = I_{B_k(0)} x_k$. $\exists \Psi$ coercive ($\Rightarrow$ l.s.c.). st.

$\mathbb{E}_{\mathbb{P}^n} (\Psi(X_k I_{B_k(0)} \circ \pi)) \leq \square$

By MCT. Fatou's (monotone..) Set $k \to \infty$.

Use the lemma in ① ii). We have:

RHS $\leq 1 + \int_0^1 Ⓜ(\sup_x |a|) + Ⓜ(\sup_x |b|)\, dt$

Note $\sum_1^n \varphi(X_k \circ Z)$ is coercive as well

$\Rightarrow$ We have tightness of $(\mathbb{P}^n)$.

3') To prove the limit satisfies FPE. It's identical with ② ii).

Under condition (A_3):

Set g is smooth cutoff to apply the method of image of measure to approxi. a.b.

Rmk: Mollifier approxi. is to obtain regularity (like under cond. (A_2)). While image of measure here can extend the local bdd cond. (A_3) to global bdd. (But it doesn't give any regularity!)

Under condition (A_4):

Approxi. by mollifier method (regular $\Rightarrow$ local bdd)