Connection with Markov

(1) Definition:

Def: $(S, \mathcal{S})$ is measure space. $\Lambda : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{M}_b^+(S) \to \mathcal{M}_b^+(S)$ has flow property if:

$\Lambda(s, t, \zeta) = \Lambda(r, t, \Lambda(s, r, \zeta))$. $\forall s \le r \le t$.

Rmk: (P_t) is Markov trans. kernel.

Let $\Lambda(s, t, \zeta)(dy) := \int_S P_{t-s}(x, dy)\zeta(dx)$

$\Rightarrow \Lambda$ has flow property (CK-equ.)

Converse is false!

Set (P_t) is a Markov trans. kernel on $(\Omega, \mathbb{P})$.

Recall: i) If S is polish. (P_t) has one-to-one correspd. with Markov process $((X_t), (\mathbb{P}_x)_{x \in S})$.

ii) $\mathbb{P}_\nu \circ (X_{t_1}, \cdots X_{t_n})^{-1} = \nu P_{t_1} P_{t_2 - t_1} \cdots P_{t_n - t_{n-1}}$.

Def: (P_t) is semigroup of Markov process X_t.

Set the dual semigroup $P_t^*: \mathcal{P}(S) \to \mathcal{P}(S)$

def by $(P_t^* \nu)(A) := \int P_t(x, A) \nu(dx)$.

<u>Rmk</u>: $(s, t, \xi) \mapsto P_{t-s}^* \xi$ also has the flow property in $\mathcal{P}(S)$.

For time-inhomogeneous Markov process. We consider the generalized model:

$\Omega_s = C([s, \infty), S)$, $Z_t^s: \Omega_s \to S$, $Z_t^s(w) = w_t$.

$\mathcal{F}_s = \sigma(X_r, r \geq s)$. $\mathcal{F}_{s,r} = \sigma(X_t, s \leq t \leq r)$.

<u>Def</u>: $(\mathbb{P}_{s,x})_{s \in \mathbb{R}^+, x \in S} \subset \mathcal{P}(S)$ is a family of measure of time-inhom. Markov process

if i) $x \mapsto \mathbb{P}_{s,x}(\Gamma)$ $\mathcal{S}$-measurable. $\forall \Gamma \in \mathcal{F}_s$

ii) (Markov property)

$$\mathbb{P}_{s,x}(Z_t^s \in B \mid \mathcal{F}_{s,r}) = \mathbb{P}_{r, Z_r^s}(Z_t^r \in B)$$

<u>Rmk</u>: It also associated with a trans. kernel $(P_{r,t})_{t \geq s}$. $P_{s,t}: S \times \mathcal{S} \to [0,1]$.

So. $P_{s,t} = P_{s,r} P_{r,t}$. defined by:

$p_{s,t}(x,A) = \mathbb{P}_{s,x}(z_t^s \in A)$

It can also be extended to semigroup

$P_{s,t} : f \mapsto \int f \, p_{s,t}(x, dy)$

$P^*_{s,t} : \nu \in \mathcal{P}(\mathbb{R}^d) \mapsto \int p_{s,t}(x,A)\, d\nu(x)$

$\Rightarrow (s,t,g) \mapsto P^*_{s,t} g$ also has flow prop.

(2) Connection:

Set $S = \mathbb{R}^d$. $a, b \in L^\infty_{loc}$. $a \geq 0$. Sym.

Prop. $(\mathbb{P}_{s,x})$ is time-inhomogeneous Markov process

with generator $A_s f(x) = \sum_{ij} a_{ij}(s,x)\, \partial_{ij} f(x)$

$+ \sum_i b_i(s,x)\, \partial_i f(x)$. where $C_c^\infty \subset \mathcal{D}(A_s)$

If a, b are regular enough. Then:

$t \mapsto \mu_t^{s,x} \triangleq P^*_{s,t} \delta_x$ is weakly conti. p.m.

solves FPE $\partial_t \mu_t = L^*_{a,b} \mu_t$. $\mu_s = \delta_x$. $t \geq s$.

Pf: WLOG. $s = 0$. $\forall f \in C_c^\infty$. We have:

$\left(\int f \, d\mu^x_{t+h} - \int f \, d\mu^x_t\right) / h \overset{CK}{=} P_{0,t}\left(\frac{1}{h}(P_{t,t+h} f - f)\right)$

Set $h \to 0$. RHS $= P_{0,t}(A_t f) = \int L_{\cdot,t} f \, d\mu_t^x$

Check the left derivative equals to RHS. integrate on $[0,t]$. $\Rightarrow \int f \, d\mu_t^x = \square$.

Next, we assume well-poseness (A_1):

$\forall (s,x)$. $\exists$ unique weakly conti. p.m. $(\mu_t^{s,x})$. solve FPE with datum $\mu_s^{s,x} = \delta_x$ st. $a_{ij}, b_i \in L^1(\mu_t^{s,x} dt)$

<u>Thm.</u> Under cond. (A_1): there $\exists$ unique time-inhomo. Markov process $\mathbb{P}_{s,x}$ with trans. kernel $P_{s,t}(x, A) = \mu_t^{s,x}(A)$. and $\mathbb{P}_{s,x}$ is law of unique weakly sol. to SDE:

$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$. $X_0 = x$.

<u>Pf</u>: Unique is from correspd of trans. kernel By superposition prin. and its unique

Thm: $\exists$ unique $(\mathbb{P}_{s,x})$ is law for weak solution of the SDE, st. $\mu_t^{s,x}$ is its one-dim marginals dist.

Next. we prove $\mathbb{P}_{s,x}$ satisfies MP.

Set $(Q_w^{(s,x),r})_{w\in\Omega}$ is r.c.p. of $\mathbb{P}_{s,x}$ w.r.t. $\mathcal{G}_{s,r}$

Apply lem in d) of last chap. with $A = \mathcal{G}_{s,r}$.

$\Rightarrow Q_{w,\geq r}^{(s,x),r} = Q_w^{(s,x),r}|_{\mathcal{B}(\Lambda_r)} \in MP_{r,\delta_{Z_r^s(w)}}$

Note that $\mathbb{P}_{r,Z_r^s(w)}$ is unique in $MP_{r,Z_r^s(w)}$

$\Rightarrow Q_{w,\geq r}^{(s,x),r} = \mathbb{P}_{r,Z_r^s(w)}$

With $\mathbb{P}_{s,x}(Z_t^s\in B \mid \mathcal{G}_{s,r}) = Q^{(s,x),r}(Z_t^s\in B)$

$= Q_{\geq r}^{(s,x),r}(Z_t^r \in B) = \mathbb{P}_{r,Z_r^s(w)}(Z_t^r\in B)$

$\mathbb{P}_{s,x}$ - a.s. (depend on B)

<u>Rmk</u>: If a,b satisfies extra conti. cond.

$\Rightarrow$ The generator (A_s) of the Markov process above has $C_c^2 \subset D(A_s)$ and:

$A_s f(x) = L_{a,b} f$.

<u>Pf</u>: $(P_{t,t+h} f - f)/h = \int f \, \mu_{t+h}^{t,x} - \mu_t^{t,x} / h$

LHS $\to A_t f$. RHS $\to \langle f, \partial_t \mu_t \rangle$

$= \langle f, L_{a,b}\mu_t \rangle$ by Def of FPE.

where $h \to 0$ along J_f.

Diagram:

i) Markov process with generator $A_s = L_{a,b}$

a.b regular enough $\xRightarrow{\text{accept}}$ weakly sol. to FPE

ii) Well-posedness of FPE $\xRightarrow{\text{accept}}$ Unique Markov process with trans. func. $= \mu$.

If a.b. regular. $\Rightarrow$ generator $A_s = L_{a,b}$.