Nonlinear FPEs

(1) Exist and Unique:

① Consider NLFPE: $\partial_t \mu_t = \partial_{ij}^2 (a_{ij}(t,\mu_t,x)\mu_t) - \partial_i (b_i(t,\mu_t,x)\mu_t)$

where $a_{ij}, b_i : \mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}^d \to \mathbb{R}^1$.

$\mu \in \mathcal{M}_b^+$. $a(t,\xi,x)$ is sym. nonnegative definite.

<u>Denote</u>: $L_{a,b,\mu} \varphi(t,x) = a_{ij}(t,\mu,x)\partial_{ij}^2 \varphi(x) + b_i(t,\mu,x)\partial_i \varphi(x)$

<u>Rmk</u>: a, b can also depend on the whole path measure $(\mu_t)_{t\geq 0}$ rather μ_t.

<u>eg</u>, (Nemytskii case)

$$a_{ij}(t,\mu,x) = \tilde{a}_{ij}(t, \frac{d\mu}{dx}(x), x).$$
$$b_i(t,\mu,x) = \tilde{b}_i(t, \frac{d\mu}{dx}(x), x).$$
$: \mathbb{R}^{+} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^1$.

$\frac{d\mu}{dx}$ is version of density of μ w.r.t dx

(Note by Lebesgue diff Thm: $\mu \in L^1_{loc} \Rightarrow$

$\lim_{r\to 0} dx(B_r(0))^{-1} \mu(B_r(x))$ exists dx-a.s.

we set $\frac{d\mu}{dx} = 0$ on limit-nonexist pt.)

Rmk: So $(\mu, y) \mapsto \frac{d\mu}{dy}(y)$ is $\mathcal{B}_{\mathcal{M}^+_{b,ac}} \otimes \mathcal{B}_{\mathbb{R}^d}$

measurable. $\mathcal{M}^+_{b,ac} = \{\mu \ll dx,\ \mu \in \mathcal{M}^+_b\}$.

In Nemytskii case. NLFPE write in density form: $\partial_t u(t,x) = \partial_{ij}(\tilde{a}_{ij}(t, u(t,x), x)u(t)) - \mathrm{div}(\tilde{b}(t, u(t,x), x)u(t))$

Where $\mu_t(x)dt = u(t,x)dxdt$. e.g. PME. p-Laplace

Df: Borel curve $(\mu_t)_{t\geq 0} \subset \mathcal{M}^+_{b,ac}$ solve the NLFPE with datum $\nu \in \mathcal{M}^+_b$ if $a_{ij}, b_i \in L^1_{loc}((0,\infty) \times \mathbb{R}^d)$ & $\forall \varphi \in C_c^\infty$, $\exists J_\varphi \subset \mathbb{R}^{\geq 0}$: $dt(J_\varphi) = 1$.

$\int \varphi\, d\mu_t = \int \varphi\, d\nu + \lim_{\tau \to 0^+} \int_\tau^t \int L_{a,b,\mu_s}\varphi\, d\mu_s ds$. $\forall t \in J_\varphi$

Rmk: For $a_{ij}, b_i \in L^1_{loc}([0,\infty) \times \mathbb{R}^d)$. $\Rightarrow$

$$\lim_{\tau \to 0^+} \square = \int_0^t \int L_{a,b,\mu_s}\varphi\, d\mu_s ds.$$

Linearized equations:

Fix Borel curve $(\mu_t) \subset \mathcal{M}^+_b$. Consider linear FPE: $\partial_t \nu_t = \partial^2_{ij}(a_{ij}(\mu_t)\nu_t) - \partial_i(b_i(\mu_t)\nu_t)$.

Where $a_{ij}(\mu_t)(t,x) = a_{ij}(t, \mu_t, x)$. $b_i(\square) = \square$.

Rmk: We called the linear FPE associated with NLFPE by μ-ℓFPE.

Note any NLFPE solution also solves μ-ℓFPE.

Lemma. If the solution is weakly conti. Then: $J_\varphi^c = \Omega$ for $\forall \varphi \in C_c^\infty$. If in addition $a_{ij}, b_i \in L^1$. then the NLFPE hold for $\forall \varphi \in C_{ub}^2$. μ_t has const mass as ν

Pf: As in linear case.

Lemma. If $(\mu_t) \subset M$ solve NLFPE with $\nu \in M_b^+$. s.t. ess sup $\mu_t(\mathbb{R}^d) < \infty$. Then $\exists$ unique weakly conti. dt-version modif. $\tilde{\mu}_t$

Pf: μ_t solves (μ-ℓFPE).

We can modify the solution of the linear FPE as before.

① Def: i) $T>0$, $M_b([0,T)\times\mathbb{R}^d)$ is n.v.s. of signed measure with finite TV. and $\|\mu\| :=$

$\sup_{Lip'} \int f d\mu$. $Lip' = \{ f \in Lip(\mathbb{R}^d, \mathbb{R}') \mid$ The Lip const. $\leq 1 \}$.

Rmk: Topo. of $\mathcal{M}_b^+([0,T] \times \mathbb{R}^d)$ is weak convergence of measure.

ii) For $V: \mathbb{R}^d \to \mathbb{R}'$. $T_0 \leq T$. $g \in C^+([0,T])$.

$\mathcal{M}_{T_0,g}(V) := \{ (\mu_t)_{[0,T_0]} \subset \mathcal{M}_b^+ \mid \int V d\mu_t \leq g(t). \forall t \leq T_0 \}$

iii) $(\mu^n) := (\mu_t^n)_{[0,T_0]} \subset \mathcal{M}_{T_0,g}(V)$ is V-convergence if $\int f d\mu_t^n \to \int f d\mu_t$. for $\forall f \in C(\mathbb{R}^d)$. s.t. $V(x)/f(x) \to +\infty$. $(|x| \to \infty)$

Assumptions:

i) $\exists V \in C^2$. s.t. $V > 0$. $\lim_{|x| \to \infty} V(x) = \infty$. s.t. $\exists \Lambda_1$. $\Lambda_2 : C^+([0,T]) \to C^+([0,T])$. for $\forall T_0 \leq T$. $g \in C^+([0,T])$. We have $L_{a,b,\nu} V(t,x) \leq \Lambda_1(g)(t) + \Lambda_2(g)(t) V(x)$ on $[0,T_0] \times \mathcal{M}_{T_0,g}(V) \times \mathbb{R}^d$.

Rmk: Next. we fix this V. $\mathcal{M}_{T_0,g}(V) := \mathcal{M}_{T_0,g}$.

ii) a_{ij}. b_i satisfy sufficient regular condition. (x-local equiconti. local bdd, V-convergence..)

iii) $\forall\, v \in \mathcal{M}_{T,q}$, $a(t, x, v_t)$ is symmetric and nonnegative definite.

Thm. Under assmpt. i) — iii). For $\mu_0 \in \mathcal{P}$. st. $V \in L^1(\mathbb{R}^d, \mu_0)$. Then: $\exists\, T_0 \leq T$. st. NLFPE has a weakly conti. p.m solution $\mu_\cdot$ on $[0, T_0]$ with datum μ_0. St. $\sup_{[0,T_0]} \int V \, d\mu_t < \infty$. Besides, if $\Lambda_1 \equiv g_1$, $\Lambda_2 \equiv g_2$, $g_i \in C^+([0,T])$.

$\Rightarrow T_0 = T$

Next, we want to apply Schauder fixed point Thm. to find the solution.

Step: i) a is sufficient smooth, nondegenerate (i.e. $\det(a) > 0$)

ii) a is sufficient smooth, degenerate.

iii) General case.

We only prove case i). i.e. Add cond. to assumpt iii) with: $\det(a(t, x, v_t)) > 0$. for $\forall\, v_t \in \mathcal{M}_{T,q}$. $\forall\, t \leq T_0$, $x \in \mathbb{R}^d$.

And assume $\forall$ cpt K. $\exists \lambda = \lambda(v, K) > 0$. s.t.

$|a(t, v_t, x) - a(t, v_t, y)| \le \lambda |x - y|$. $\forall x, y \in K$. $t \le T_0$.

And also $\exists C_i = C_i(v)$. s.t.

$|\sqrt{a(t, v_t, x)}\, \nabla V(x)| \le C_1 + C_2 V(x)$. $\forall (t, x) \in [0, T_0] \times \mathbb{R}^d$.

<u>Rmk</u>: Under the condition with assmpt i) - iii).

We have existence of unique weakly conti. p.m. solution $\mathcal{G} = \mathcal{G}(v)$ to v-LFPE on $[0, T_0]$. with μ_0 for $v \in \mathcal{M}_{T_0, g}$.

We def $\mathcal{Q}: \mathcal{M}_{T_0, g} \to \mathcal{M}_b([0, T_0] \times \mathbb{R}^d)$ by $\mathcal{Q}(v) = \mathcal{G}(v)$. ($\mathcal{Q}$ depend on T_0, g).

<u>Def</u>: $\mathcal{N}_{T_0, g} := \{(\mu_t) \in \mathcal{M}_{T_0, g} \mid \forall \varphi \in C_c^\infty(\mathbb{R}^d).$

$$\left|\int \varphi \, d\mu_t - \int \varphi \, d\mu_s\right| \le \sup_{\substack{[0,T_0] \times \\ \mathcal{M}_{T_0,g} \times \mathbb{R}^d}} |L_{a,b,v} \varphi(t, x)| \, |t - s| \}$$

<u>Lemma</u>. $\forall (\mu^n) \subset \mathcal{N}_{T_0, g}$ has weakly convergence subseq (μ^{n_k}). $\xrightarrow{w} \mu \in \mathcal{N}_{T_0, g}$. s.t. $\forall t \le T_0$.

$\mu_t^{n_k} \xrightarrow{w} \mu_t$.

Rmk: Note $\mu^n \xrightarrow{w} \mu \Leftrightarrow \mu^n_t \xrightarrow{w} \mu_t\ \forall t$. Since

$(\mu^{n_k}) \xrightarrow{w} \mu \Leftrightarrow \forall \varphi \in C_c^\infty([0,T]\times\mathbb{R}^d)$.

$\int_{[0,T]\times\mathbb{R}^d} \varphi(t,x)\mu^n_t(x) \to \int \varphi(t,x)d\mu_t$.

Pf: Consider $(t_k) \overset{dense}{\subset} [0,T_0]$. by diagonal argument. $\exists (\mu^{n_k}_t)$ is tight for $\forall t \in (t_k)$.

Then, we prove it also holds for $\forall t \in [0,T_0]$ by def of $N_{T_0,q}$.

$(\mu^{n_k}) \xrightarrow{w} \mu$ is from DCT. (i.e. $\int h(t,x)d\mu^{n_k}_t$ is t-uniformly bdd).

Cor: $N_{T_0,q}$ is convex. cpt. in $\mathcal{M}_b([0,T]\times\mathbb{R}^d)$.

Pf: Convex is from $\mathcal{M}_{T_0,q}$ is convex.

And $\mathcal{M}_b(\cdot\cdot)$ is n.v.s. So:

seq. cpt $\Leftrightarrow$ cpt. We use Lem. above.

Lemma 2: $\mu^n \xrightarrow{w} \mu$ in $N_{T_0,q} \Rightarrow \mu^n \xrightarrow{V} \mu$.

Pf: i) $(\mu^n_t) \to \mu_t\ \forall t \in [0,T_0]$ follows from Lemma' above. (By contradict.:

if $\exists t_0, (n_\ell)$. s.t. $\mu_{t_0}^{n_\ell} \xrightarrow{w} \tilde{\mu} \neq \mu_{t_0}$

$\exists (n_{\ell k}) \subset (n_\ell)$ $\mu_{t_0}^{n_{\ell k}} \xrightarrow{w} \mu_{t_0}$. Contradict!)

2) Set $h = f(x)/V(x)$. $\Rightarrow h \in C_0$.

So: $\exists \varphi \in C_c^\infty$. s.t. $|h-\varphi|_\infty < \varepsilon$.

So: $|\int f \, d\mu_t^n - \int f \, d\mu_t| \leq$

$|\int \varphi V d\mu_t^n - \int (\varphi V) d\mu_t| + 2\varepsilon |g|_\infty \xrightarrow{1)} 0$.

<u>Lemma'</u>: If $Q(N_{T_0,q}) \subset N_{T_0,q}$ for some $T_0 \leq T$. &

$q \in C^+([0,T])$. Then Q is conti. on $N_{T_0,q}$

<u>Pf</u>: For $(\mu^n) \to \mu$ in $N_{T_0,q}$. Set $g^n :=$

$Q(\mu^n) \in N_{T_0,q}$.

By Lemma' above. $\exists (g^{n_k}) \xrightarrow{w} g \in N_{T_0,q}$.

Note: $\int_{\mathbb{R}^d} \varphi \, d g_t^{n_k} - \int_{\mathbb{R}^d} \varphi \, d\mu_0 = \int_0^t \int_{\mathbb{R}^d} L_{a,b,\mu^k} \varphi \, dg_s^{n_k} ds$ for $\varphi \in C_c^\infty$.

From assumption ii). We have Ascoli. Thm

hold locally for $a_{ij}(t, \mu_t^{n_k}, x)$. $b_{ij}(\cdots)$.

Let $k \to \infty$. (since supp(φ) is cpt.)

with DCT. (since $|L_{a,b,\mu^n}\varphi| \le \Lambda(T_0, q, \varphi)$)

Σ : RHS $\to \int_0^t \int_{\mathbb{R}^d} L_{a,b,\mu}\varphi \, d\zeta \, ds$.

which means $Q(\mu_t) = \zeta_t$.

By subseq convergence Thm $\Rightarrow Q(\hat{\mu}^n) \to Q(\mu)$

<u>Lemma.</u>[4] $V \in N_{T_0,q}$, $\zeta = Q(V)$. Then : $\forall t \in [0, T_0]$.

$\int V d\zeta_t \le S[q](t) + R[q](t) \int V d\mu_0$. where

$R[q](t) = e^{\int_0^t \Lambda_2(q)(s) ds}$. $S[q](t) = R[q](t) \int_0^t \Lambda_1[q](s) \, ds$.

<u>Pf</u>: Set $\eta_m \in C^\infty(\mathbb{R}^+)$. $0 \le \eta_m' \le 1$. $\eta_m'' \le 0$.

$\eta_m(x) = x$ if $x \le m-1$. $\eta_m(x) \equiv m$. if $x > m$

Set $\varphi(x) = \eta_m \circ V(x) - m \in C_0^2$

$L_{a,b,v}\varphi = \eta_m'(V(x)) L_{a,b,v} V(t,x) + \eta_m''(V(x))\, a(t, V_t, x) \nabla V(x) \cdot \nabla V(x)$. $(*)$

since ζ_t solves V-LFPE. we have:

$\int_{|x| \le m-1} V d\zeta_t \le \int_{\mathbb{R}^d} \varphi \, d\zeta_t \overset{FPE}{=} \square$

$\overset{(*)}{\le} \int V d\mu_0 + \int_0^t \int_{|V| \le m} \eta_m'(V(x)) L_{a,b,v} V(s,x) \, d\zeta_s(x) \, ds$

Since $\eta^m_{m} \leq 1$. with assupt. i). :

$$RHS \leq \int_1^t \Lambda_1[g](s) + \Lambda_2[g](s) \int_{|V| \leq m} V \, d\xi_s \, ds$$

Let $m \to \infty$. (MCT). Apply Gronwall's:

$$\int_{\mathbb{R}^d} V \, d\xi_t \leq S[g](t) + R[g](t) \int V \, d\mu_0.$$

Cor. $\exists T_0 \leq T$. $g \in C^+([1,T])$. s.t. $\Phi(N_{g,T_0}) \subset N_{g,T_0}$. Besides, if $\Lambda_1, \Lambda_2 \equiv$ const in $C^+([1,T])$, then $T_0 = T$.

Pf: For $\forall$ fix g. $S[g](t) \to 0$. & $R[g](t) \to 1$ $(t \to 0)$.

Set $g = 2 \int V \, d\mu_0 + 1$.

Choose $T_0 = T_0(g)$, s.t. On $[1,T_0]$ $S[g](t) \leq 1$. $R[g](t) \leq 2$

$\Rightarrow \int V \, d\xi_t \leq g(t)$ on $[1,T_0]$.

for $\forall \nu \in N_{T_0,g}$. $\xi = \Phi(\nu)$.

We can set $g(t) = \sup\limits_{[1,T]} (S(r) + R(r) \int V \, d\mu_0)$ if Λ_1, Λ_2 are const. So $S[\cdot](t)$. $R(\cdot)(t)$ are indept. of choice of g.

Return to the proof:

Since $\exists T_0, q$. s.t. $\Phi: N_{T_0,q} \to N_{T_0,q}$ is conti. $N_{T_0,q}$ is convex. cpt. $\subset M_b([0,T]\times\mathbb{R}^d)$

Apply Schauder's fixed pt. Thm. $\exists\ \Phi(\nu)=\nu$.

(2) <u>Superposition principle:</u>

① consider McKean - Vlasov SDEs / DDSDEs:

$dX_t = b(t, X_t, \mathcal{L}_{X_t})dt + \sigma(t, X_t, \mathcal{L}_{X_t})dB_t$.

<u>Rmk:</u> We can fix $\mu_t = \nu_t$ inside b and σ to get a ν-lSDE:

$dX_t = b(t, X_t, \nu_t)dt + \sigma(t, X_t, \nu_t)dB_t$

And solution for DDSDEs also solves μ-lSDE where $\mu = \mathcal{L}_X$.

<u>Prop.</u> If X is weak solution for DDSDEs above. Then: $(\mu_t) := \mathcal{L}_{X_t}$ is weakly conti. p.m. solution to NLFPE with coeff. b and $a = \frac{1}{2}\sigma\sigma^T$.

Pf: By Itô's formula.

(A0) $b(t,\cdot,\cdot)$ is continuous on $\mathcal{P}_p \times \mathbb{R}^d$ for all $t \geqslant 0$.

(A1) $\exists\, K_1, K_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ non-decreasing such that for all $t \geqslant 0, \zeta, \nu \in \mathcal{P}_p, x, y \in \mathbb{R}^d$

$$|\sigma(t,\zeta,x) - \sigma(t,\nu,y)|^2 \leqslant K_1(t)|x-y|^2 + K_2(t)\mathcal{W}_p(\zeta,\nu)^2.$$

(A2) $2(b(t,\zeta,x) - b(t,\nu,y)) \cdot (x-y) \leqslant K_1(t)|x-y|^2 + K_2(t)\mathcal{W}_p(\zeta,\nu)|x-y|$.

(A3) b is bounded on bounded sets in $\mathbb{R}_+ \times \mathcal{P}_p \times \mathbb{R}^d$, and

$$|b(t,\zeta,0)|^p \leqslant K_1(t)\big(1 + \zeta(|\cdot|^p)\big),$$

where $\zeta(|\cdot|^p) = \int_{\mathbb{R}^d} |x|^p \, d\zeta(x)$.

Theorem 3.3.4 (Thm.2.1 from [22]). *Assume there is $p \geqslant 1$ such that (A0)-(A3) are satisfied. (If $p < 2$, additionally assume $K_2 = 0$.) Then for every initial datum $\mu_0 \in \mathcal{P}_p$, the DDSDE has a unique weak solution $X(\mu_0)$ with $\mathcal{L}_{X_t} \in \mathcal{P}_p$ for all $t \geqslant 0$. Moreover, if $\mu_0 \in \mathcal{P}_q$ for $q \geqslant p$, then* can be strong.

$$\mathbf{E}\Big[\sup_{t\in[0,T]} |X(\mu_0)_t|^q\Big] < \infty, \quad \forall T > 0.$$

[*Finally, there is $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ non-decreasing such that*

$$\mathcal{W}_p\big(\mathcal{L}_{X(\zeta)_t}, \mathcal{L}_{X(\nu)_t}\big)^p \leqslant \mathcal{W}_p(\zeta,\nu)^p e^{\int_0^t \psi(r)dr}, \quad \forall t > 0.$$]

③ Thm. (Superposition prin.)

If (μ_t) is weakly conti p.m solution to NLFPE. s.t. $a, b \in L((0,T)\times\mathbb{R}^d; \mu_t dt)$

Then $\exists$ weak solution X to DDSDE s.t. $\mathcal{L}_{X_t} = \mu_t$. $\forall t \geq 0$.

Rmk: i) There's still no extra regularity assumption. It can be applied on Nemytskii type NLFPE.

ii) Integrable cond. can be weakened as before.

Pf: (μ_t) also solves μ-LFPE.

$\exists$ weak solution to μ-LSDE s.t.

$\mathcal{L}X_t = \mu_t$. $\Rightarrow$ X_t also solves DDSDE.

Cr. At most one weak solution to DDSDE with initial datum ζ $\Rightarrow$ At most one p.m sol. associate NLFPE with datum ζ. s.t. $a_{ij}, b \in L^1(\lambda_{\mu_t} dt)$.

Pf: Any 2 solutions of NLFPE can be be lifted as solution of DDSDE as linear case $\Rightarrow$ 1-dim marginal dist. equals in particular.

prop. For $\mu_0 \in \mathcal{P}$. if:

i) NLFPE has unique weakly conti. p.m. solution μ with datum μ_0.

ii) μ-LFPE has unique weakly conti. p.m. solution for $\forall$ datum (s, δ_x).

Then weak solution for DDSDE with datum μ_0 is unique.

Pf: If X, Y are 2 weak solution to DDSDE with datum μ_0.

Set $\mu^1_\cdot := \mathcal{L}_{X_\cdot}$, $\mu^2_\cdot := \mathcal{L}_{Y_\cdot}$

We have $\mu = \mu_1 = \mu_2$ from i)

And from ii), μ-LFPE also have unique solution. and X, Y also satisfy μ-LSDE. $\Rightarrow (X_t)_{t\geq 0} \overset{\mathcal{L}}{\sim} (Y_t)_{t\geq 0}$.

With Mart. problem:

i) Uniqueness of DDSDE for any δ_x, $x \in \mathbb{R}^d$ can't imply uniqueness for any initial datum μ_0:

Note if X is solution to DDSDE with datum μ_0. The disintegration family $(Q_x)_{x\in}$ of $\mathcal{L}_X$ isn't the law of solution to DDSDE:

$dX_t = b(t, X_t, \mathcal{L}_{X_t})dt + \sigma(t, X_t, \mathcal{L}_{X_t})dB_t$, $X_0 \sim \delta_x$.

since $\mathcal{L}_{\delta_x}^t \neq \mathcal{L}_{X_t^{\mu_0}}$ unless $\mu_0 = \delta_x$.

Q_x rather solves $\mathcal{L}_{X^{\mu_0}}$ - LSDE with the initial dist. $= \delta_x$. (i.e. fix μ_0-variable).

ii) Unlike linear case, Mart. Problem doesn't hold. i.e. well-posed solution $(\mathbb{P}_x)$ for DDSDE isn't Markov process:

Since linear case heavily depends on the stability of MP for disintegration family. But this fails as in i).

But if we fix $\mu_t^x = \mathbb{P}_x \circ Z_t^{-1}$. And assume μ^x-LSDE is weakly well-posed

$\Rightarrow \exists (\tilde{\mathbb{P}}_y^x)_{y \in \mathbb{R}^d}$ unique solution to the μ^x-LSDE with datum δ_y. It's also family of Markov process.

And note $\tilde{\mathbb{P}}_x^x = \mathbb{P}^x$. i.e. $(\mathbb{P}^x)$ can be embedded in family of Markov process

Program:

1) Existence of weakly conti. p.m. s.l. to NLFPE:

$\exists$ V. lyapounov. + conti. of n.b. + $a \geq 0$

Rmk: i) It remains work for Nemytskii.

ii) If Δ_1, Δ_2 dominant $\equiv$ const in $C \geq 0$

$\Rightarrow$ Existence interval $= [0, T]$.