Flow Selection

Denote $S_{s,\zeta} := \{\mu$ weakly cont. p.m. solves NLFPE with (a,b), datum ζ from $t=s\}$.

Next, we want to know:

Is there $\mu^{s,\zeta} \in S_{s,\zeta}$, s.t. $\mu^{s,\zeta}$ satisfies flow property, i.e. $\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}$, $\forall s \le r \le t$. (called solution flow) and for $\forall \zeta \in \mathcal{P}$?

(1) <u>Crandall-Liggett Semigroup Method:</u>

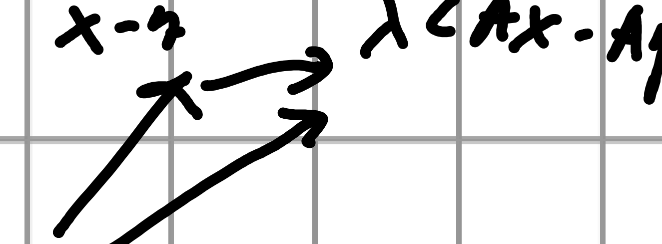
It's a way to construct $\mu^{s,\zeta}$ explicitly.

<u>Def:</u> $(X, |\cdot|)$ is Banach space.

i) $A: D(A) \subseteq X \to X$ is accretive if

$|x-y| \le |(\lambda A+I)(x) - (\lambda A+I)(y)|$, for

$\forall \lambda > 0$, $\forall x, y \in D(A)$.



<u>Rmk:</u> i) $(A, D(A))$ is accretive $\iff$ it holds for some $\lambda > 0$.

ii) We have $I+\lambda A$ is injective

& $(I+\lambda A)^{-1}: R(I+\lambda A) \to D(A)$

satisfies Lipschitz cond.

iii) Accretive is some kind of

mono. property in Banach space.

ii) Accretive op. A is called m-accre.

if $R(I+\lambda A) = X$. for $\forall \lambda > 0$.

Rmk: m-accre. $\Longleftrightarrow$ accre. & $R(I+\lambda A)$

$= X$ for some $\lambda > 0$.

iii) $(A, D(A))$ is called w-quasi m-accre.

if $(A+wI, D(A))$ is m-accre.

iv) $(A, D(A))$ is dissipative / m-dissipative /

quasi. m-disp. if $-A$ is accre / m-....

Recall Cauchy problem $y'(t) = Ay(t)$. $y(0)=y_0$.

which is understand in classical sense. &

solve it pointwise. This is theory of strong

solutions. Next we intro another view:

Def: i) ε-discretization of $[0,T]$ is partition $P^\varepsilon(t_0, \cdots t_N)$. $0 = t_0 \le \cdots t_N = T$. s.t. $t_i - t_{i-1} \le \varepsilon$

ii) A $P^\varepsilon(t_0 \cdots t_N)$ solution to Cauchy problem above on $[0,T]$ is piecewise const func. $z^\varepsilon : [0, t_N] \to X$. s.t. value z_i^ε on $(t_{i-1}, t_i]$ is def recursively: $z_0^\varepsilon = y_0$

$z_i^\varepsilon = (t_i - t_{i-1}) A z_i^\varepsilon + z_{i-1}^\varepsilon$. $1 \le i \le N$

rmk: It discretizes the differential.

iii) Set of ε-approxi. solution collects all ε-$P(t_0 \cdots t_N)$ solutions.

Def: A mild solution to the Cauchy problem above on $[0, \infty)$ is $y \in C([0,\infty), X)$ s.t. $\forall \varepsilon > 0$. $\forall T > 0$, $\exists$ ε-approxi solu. z_ε. on $[0,T]$. s.t. $\sup_{[0,T]} |y - z_\varepsilon| \le \varepsilon$.

Thm. (Crandall - Liggett)

$(A, D(A))$ is ω-quasi. m-dissipative, $y_0 \in \overline{D(A)}$. Then the Cauchy problem has unique mild solution $y_t(y_0)$ on $\mathbb{R}^{\ge 0}$.

given by $y_t(y_0) = \lim_{n\to\infty} (I - \frac{t}{n}A)^{-n} y_0$.

where the limit is locally uniform on t.

<u>Cr.</u> $S(t, y_0) := y_t(y_0) \overset{\Delta}{=} e^{tA} y_0$ is

a semigroup. Satisfies $S_{t+s}(y_0)$

$= S_t \circ S_s(y_0)$. i.e. flow property.

<u>e.g.</u> generalized PME: $\partial_t u = \Delta\beta(u) - \mathrm{div}(D B(u) u)$

under $\beta \in C^2(\mathbb{R}^1)$. D, B bdd.

Set $A_0 : D(A_0) \subset L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$. defined by

$A_0 y = \Delta\beta(y) - \mathrm{div}(DB(y) y)$

$\Rightarrow D(A_0) = \{ y \in L^1(\mathbb{R}^d) \mid \beta(y) \in L^1_{loc}.$

$\Delta\beta(y) - \mathrm{div}(DB(y) y) \in L^1(\mathbb{R}^d) \}$

We can prove:

i) $R(I - \lambda A_0) = L^1(\mathbb{R}^d)$. $\forall \lambda > 0$

ii) $\exists$ restriction $(A, D(A))$. s.t. $D(A) \subset D(A_0)$

and A is dissipate on $L^1(\mathbb{R}^d)$.

iii) $\overline{D(A)} = L^1(\mathbb{R}^d)$

So apply the Thm above. we have:

$\exists$ unique mild solution $u_t(u_0)$. for that $u'(t) = Au(t)$, $u(0) = u_0$ for $\forall u_0 \in L^1(\mathbb{R}^d)$, which has flow property in $L^1(\mathbb{R}^d)$.

<u>Rmk:</u> i) We can also show in this case: $t \mapsto u(t,x)dx$ is weakly conti. sol. to Nemytskii-type NLFPE and $|u_t|_{L^1} = |u_0|_{L^1}$. $\forall t > 0$

ii) $u(u_0)$ isn't necessary solution for $A_0 u = u'(t)$. since we restrict A_0.

iii) Uniqueness is in sense of "mild".

(2) <u>Flow Selection:</u>

① <u>Remark:</u> i) $SP_{s,\zeta} := \{ \mu$ is s.p.m, vaguely conti. | μ solve NLFPE $(a.b)$ with datum $(s,\zeta)\}$

ii) $SP_s := \bigcup SP_{s,\zeta}$, $\zeta \in SP$

<u>Def:</u> A family $\{A_{s,\zeta}\}_{s\geq 0, \zeta \in SP}$. $A_{s,\zeta} \subset SP_s$. is flow admissible if:

i) $(\mu_t)_{t\geq r} \in A_{r,\xi} \Rightarrow (\mu_t)_{t\geq s} \in A_{s,\mu_s}$. $s\geq r$.

ii) $(\mu_t)_{t\geq r} \in A_{r,\xi}$ $(\eta_t)_{t\geq s} \in A_{s,\mu_s}$. $\Rightarrow$

$$(\mu \circ_s \eta)_t := \begin{cases} \mu_t & r\leq t\leq s \\ \eta_t & t\geq s \end{cases} \in A_{r,\xi}$$

Next, we denote $A_s := \{\xi \in \mathcal{SP} \mid A_{s,\xi} \neq \emptyset\}$ and call (s,ξ) is admissible if $\xi \in A_s$

<u>eg.</u> i) $A_{s,\xi} = \mathcal{SP}_{s,\xi}$

ii) $A_{s,\xi} = \begin{cases} \emptyset, & \text{if } \xi \notin \mathcal{P} \\ S_{s,\xi}, & \text{if } \xi \in \mathcal{P} \end{cases}$

iii) Set $\mathcal{SP}^{\ll}_{s,\xi} := \{(\mu_t)_{t\geq s} \in \mathcal{SP}_{s,\xi} \mid d\mu \ll dx\}$

Then for $\mathcal{SP}^{\ll} \subseteq \mathcal{M} \subseteq \mathcal{SP}$. We set

$$A_{s,\xi} = \begin{cases} \mathcal{SP}^{\ll}_{s,\xi}, & \xi \in \mathcal{M} \\ \emptyset, & \xi \notin \mathcal{M} \end{cases}$$

$\Rightarrow$ i), ii), iii) are all flow admissible.

<u>Rmk:</u> i) From ii), we know that if $|S_{s,\xi}| = 1$, for $\forall (s,\xi)$ admissible, and $\xi \in \mathcal{P}$. Then $\mu^{s,\xi} \in S_{s,\xi}$ has flow property, from prop. i). Since $(\mu^{s,\xi})_{t\geq r}$

will be a new element otherwise.

And it doesn't hold if a_t, b_t depend not only on (μ_t, x) but $((\mu_r)_{r\le t}, x)$

ii) Case (iii) is common in Nemytskii

Next, set τ_v is vaguely convergence topo on SP while τ_{pt} is point-wise topo.

Thm. (H, τ) is Hausdorff. st. $H \subseteq SP$. $\tau \supseteq \tau_v$

Assume $\{A_{s,\zeta}\}_{s\ge 0, \zeta\in SP}$ is flow admissible.

If $A_{s,\zeta}$ is cpt in $(C_s H, \tilde{\tau})$. $\tilde{\tau} \supseteq \tau_{pt}$

Then $\exists$ solution flow to NLFPE in $\{A_{s,\zeta}\}$

Def: i) $\zeta : \mathbb{N} \times \mathbb{Q}^{\ge 0} \to \mathbb{N}_0$ is one ordering

ii) For $s \ge 0$. $(m_k^s)_{k\in\mathbb{N}_0} \subseteq \mathbb{N}_0$ is the enumeration of $\mathbb{N}_0 \times \mathbb{Q}^{\ge s}$. w.r.t ζ.

i.e. $m_k^s = \zeta(n', k')$. where (n', k') is the $k+1$-th element in $\mathbb{N}_0 \times \mathbb{Q}^{\ge s}$.

Rmk: $(m_k^s)_k \subset (m_k^r)_k$ for $s \ge r$.

Lemma. i) H is Haus. $\Rightarrow (C_s H, \tau_{pt})$ is Haus

iii) $\exists$ countable measure-separating family $(f_k)_{k\geq 0} \subseteq C_c(\mathbb{R}^n)$, i.e. for $\mu^i \in \mathcal{M}_b^+$,

$\mu^1 \neq \mu^2 \iff \exists k$. s.t. $\int_{\mathbb{R}^n} f_k \, d\mu^1 \neq \int f_k \, d\mu^2$.

Pf: iii) $\{[\vec{a},\vec{b}]\}_{\vec{a},\vec{b}\in\mathbb{Q}}$ is basis of $\mathbb{R}^n$.

We chose $(f^k_{\vec{a},\vec{b}}) \xrightarrow{n} \mathbb{1}_{[\vec{a},\vec{b}]}$

And collect all such function.

Pf of Thm:

Set $\mathcal{H} := \{h_n\}$ the countable separa. family.

ζ is a ordering. let (S,ζ) is admissible.

Set $G_0^{s,\zeta} : C_s\mathcal{M} \to \mathbb{R}^1$, $(\mu_t)_{t\geq 1} \mapsto \int_{\mathbb{R}^n} h_{n_{m_0^\zeta}} \, d\mu_{\zeta_{m_0^\zeta}}$

$\mathcal{K}_0^{s,\zeta} = \sup_{A_{s,\zeta}} G_0^{s,\zeta}(\mu)$.

$\mathcal{M}_0^{s,\zeta} = (G_0^{s,\zeta})^{-1}(\mathcal{K}_0^{s,\zeta}) \cap A_{s,\zeta}$.

Note $A_{s,\zeta}$ is cpt by assupt. and nonempty

$\Rightarrow \mathcal{M}_0^{s,\zeta}$ is cpt. nonempty in $C_s\mathcal{M}$.

Iteratively, replace "0" by "$k+1$" and $A_{s,\zeta}$ by $\mathcal{M}_k^{s,\zeta}$ above. We got a nested

Seq $\{\mathcal{M}_k^{s,\xi}\}_{k\geq 0}$. Since $C_{s}\mathcal{U}$ is Hausdorff

$\Rightarrow \bigcap_k \mathcal{M}_k^{s,\xi} := \mathcal{M}^{s,\xi} \neq \emptyset$.

For $(\mu^{(i)}) \in \mathcal{M}^{s,\xi}$. $\Rightarrow \int_{\mathbb{R}^d} h_{n_{r_k^i}} d\mu^{(1)}_{q_{m_k^i}} = \square^{(2)}$

Note $\{(n_k^i, q_{m_k^i})\}_{k\geq 1} = \mathbb{N}\times\mathbb{Q}^{\geq s}$. So:

$\int h_n d\mu^{(1)}_q = \square^{(2)}$, $\forall n, q \in \mathbb{N}\times\mathbb{Q}^{\geq s}$.

$\Rightarrow \mu^{(1)}_q = \mu^{(2)}_q$. $\forall q \in \mathbb{Q}^{\geq s}$. from choice of $\mathcal{H}$

So: $\mu^{(1)} = \mu^{(2)}$ since they're conti. i.e.

$\mathcal{M}^{s,\xi} = \{\mu^{s,\xi}\}$ is singleton.

Next, we show $\{\mu^{s,\xi}\}$ has flow prop.

For $0 \leq s < r$. ~~Set~~ $\mathcal{M}^{r,\mu_r^{s,\xi}} = \{(\gamma_t)_{t\geq r}\}$.

prove: $\gamma_t = \mu_t^{s,\xi}$. $\forall t \geq r$.

~~Set~~ $\eta := \mu^{s,\xi}$ or $\gamma \in A_{s,\xi}$.

We want to prove: $\int h_{n_{m_k^i}} d\mu^{s,\xi}_{q_{m_k^i}} = \int h_{\square}$

$d\eta_{q_{m_k^i}}$ as above.

For $k=0$. "$\geq$" is from max property of $\mu^{s,\xi}$

"$\leq$" is by arguing $q_{m_0^i} \in (s,r]$ or $q_{m_0^i} \geq r$

(then $q_{m_0^s} = q_{m_0^r}$). which uses max property of γ in $A_{r,\mu^{s,\xi}_r}$.

for $n = k$. Consider $q_{m_k^s} \in (s, r]$ for "$\le$" part and $q_{m_k^s} \ge r$. $q_{m_{k-1}^s} \in (s, r]$ (then $m_k^s = m_{k-1}^r$) or $q_{m_{k-1}^s} \ge r$ (then $m_k^s = m_k^r$. $m_{k+1}^s = m_{k+1}^r$)

for another side "$\ge$", using max of γ.

<u>prop.</u> Under the conditions above. i) $\Leftrightarrow$ ii) :

i) $\exists$ at most one solution flow to NLFPE in $\{A_{s,\xi}\}_{s \ge 0,\, \xi \in SP}$

ii) $|A_{s,\xi}| \le 1$. $\forall s \ge 0$. $\xi \in SP$.

<u>Pf:</u> ii) $\Rightarrow$ i) is trivial. For i) $\Rightarrow$ ii) :

If $\exists$ ad. $(s', \xi') \in [0, T) \times SP$. st. $|A_{s',\xi'}| \ge 2$. Assume $\{\mu^{s,\xi}\}$ is the solution flow on all ad. (s, ξ) w.r.t. enumeration $\mathcal{S}$ and separate family $\mathcal{H} = \{h_n\}$. st. $\mathcal{H} = -\mathcal{H}$. It means : $\exists\, \gamma \ne \mu^{s,\xi}$. $\gamma \in A_{s',\xi'}$.

S_1: $\exists q \in \mathbb{Q}^{\geq 1}$. s.t. $\mu_q^{s:1'} \neq \zeta_q$.

By max prop. of $\mu^{s:1'}$. $\exists h \in \mathcal{H}$. s.t.

$\int h \, d\mu_q^{s:1'} > \int h \, d\zeta_q$.

Next, we want another enumeration

S' to index $-h$ as same as h

so reverse the inequ. to contradict.

i.e. $(h_{n m_0'}, q m_0') = (-h, q)$.

S_2: $\exists$ flow $(\eta^{s,t})$ w.r.t $\mathcal{H}$ and S'.

With max prop. of $\eta^{s,t}$ in $A_{s:1}$.

$\Rightarrow \int -h \, d\eta_q^{s:1'} = \int -h \, d\zeta_q$.

But with inequ. above. $\eta^{s:1'} \neq \mu^{s:1'}$. both

flow of NLFPE. contradict!

③ Application:

Denote τ_{co} is cpt-open topo. on $C(X, Y)$.

generated by $\{ \{ f \in C(X, Y) \mid f(K) \subset O \} \mid K$ cpt

and O open. $\}$

Prop. (Arzela - Ascoli.)

I is interval. (Y, d) is metric space.

$\mathcal{G} \subseteq C(I, Y)$ is relatively cpt w.r.t.

τ_{co} $\overset{(*)}{\iff}$ $\{f(t) \mid f \in \mathcal{G}\}$ is relatively cpt

in Y $^{(**)}$ and $\mathcal{G}$ is equiconti.

Rmk: Under Y is metrizable.

i) τ_{lu}, locally uniform topo. equi. with τ_{co} on $C(Y)$. So it's indept. of choice of compatible metric on Y.

ii) Equiconti. of $\mathcal{G} \subset C(Y)$ usually depend on choice of metric of Y. But Ascoli asserts equiconti. of pointwise relat-cpt set is indept of choice of compatible metric. (Since $(*)$. $(**)$ are unrelated!)

i) Linear FPE:

Consider $\partial_t \mu_t = \partial_{ij}^2 (a_{ij}(t,x) \mu_t) - \partial_i (b_i(t,x) \mu_t)$

Assumpt :

A1) $\int_0^T \sup_x |a_{ij}| + |b_i| \, dt < \infty$. $\forall T > 0$. $\forall i, j$.

A2) $x \mapsto a_{ij}$. b_i are conti. for dt - a.e.

prop. Under A1. A2. If $SP_{s,\zeta} \neq \emptyset$. $\forall s \geq 0$ and $\zeta \in \mathcal{P}$. Then $\exists$ solution flow to the FPE in $A_{s,\zeta} = SP_{s,\zeta}$.

Rmk: Recall for $\zeta \in \mathcal{P}$. we have $SP_{s,\zeta}$ is actually $S_{s,\zeta}$.

Pf: Set $(H, \tau) = (SP, \tau_v)$

Next, we want to prove $A_{s,\zeta}$ is cpt in $C(_sH, \tau_{(w)})$. i.e. it's closed pointwise relative cpt & equiconti.

1) Pointwise relatively cpt is from (SP, τ_v) is cpt. metrizable.

2) Closedness: (Then by 1) $\Rightarrow$ cpt.) Note by Rmk i) above. $(SP, \tau_v) = (SP, \tau_{(w)})$. So for $\tau_{(w)}$-Conv.

Seq $(\mu^{(n)}) \subset A_{s,\zeta} \to \mu \in C_s SP$.

We prove $\mu \in A_{s,\zeta}$

It follows from A1) and DCT:

$\int_s^t \int_{\mathbb{R}^d} L_{a,b} \varphi \, d\mu_r^{(n)} \underset{\doteq g_n(r)}{dr} \to 0$ $\left(\begin{array}{l} \text{Also with} \\ \mu_r^{(n)} \xrightarrow{v} \mu_r . \end{array} \right)$

3) Equicontinuity:

By Rmk ii). We can choose d

is τ_v-compatible metric in SP.

$d_v(\zeta_1, \zeta_2) := \sum_k 2^{-k} |\int f_k d\zeta_1 - \int f_k d\zeta_2| \wedge 1$. Where $\mathcal{G} := [f_k]_k \subseteq C_c^2$

fixed and $\overline{\mathcal{G}}^{2n} \supseteq C_c(\mathbb{R}^d)$

By A1). use def of solution to

FPE: $d_v(\mu_{t_1}, \mu_{t_2}) \leq \sum 2^{-k}$.

$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |L_{a,b} f_k(t)| \, d\mu_t dt \wedge 1$

$\lesssim |t_1 - t_2|$. indept of μ.

Rmk: The final estimate also indept of $\zeta \in SP$. So $\cup_{SP} A_{s,\zeta}$ is also relatively cpt in $C_s M_{\tau_{\ln}}$

ii) Nonlinear FPE :

Assmpt.

A_1') a_{ij}, b_i are bdd on $[0,T] \times SP \times \mathbb{R}^d$. $\forall T > 0$

A_2') $x \mapsto a_{ij}, b_i$ are conti. $\forall \zeta \in SP$. dt-a.e.

A_3') If $\zeta_n \xrightarrow{v} \zeta$ in SP. then $a_{ij}(t, \zeta_n, x)$, $b_i(\cdots)$

$\to a_{ij}(t, \zeta, x)$, $b_i(t, \zeta, x)$. local unif on x.

Rmk : i) It pays more price (A_3') for the nonlinearity

ii) A_3') excludes the Nemyskii case :

since $\mu \mapsto a_{ij}(t, \mu, x) = \tilde{a}_{ij}'(t, \frac{d\mu}{dx}(x), x)$ isn't conti. on τ_v or τ_w.

since $(\mu, \tau_{v/w}) \mapsto \frac{d\mu}{dx}$ isn't conti.

(i.e. weak/vague convergence $\not\Rightarrow$ converg. of density func.)

iii) Under A_1') $-$ A_3') above. $SP_{s,\zeta} = S_{s,\zeta}$ for $\zeta \in \mathcal{P}$.

Next, replace A_i) with A_i') above and

Consider in NLFPE. We prove $\exists$ solution flow for NLFPE in $\{A_{s,y} = \{P_{s,y}\}_{s\geq 0, y\in SP}$.

Pf: $(M, \tau) = (SP, \tau_w)$. Pointwise relatively cpt and equiconti. are identical as above. from A_1').

For closedness: if $(\mu^{(n)}) \subset A_{s,y} \xrightarrow{\tau_w} (\mu) \in C_s SP$.

prove: $\int_s^t \int_{\mathbb{R}^d} L_{a,b,\mu_r^{(n)}} \varphi \, d\mu_r^{(n)} dr \xrightarrow{n\to\infty} \square$.

Since $\int_{\mathbb{R}^d} L_{a,b,\mu_r^{(n)}} \varphi \, d\mu_r^{(n)} = \langle L_{a,b,\mu_t^{(n)}} \varphi(t), \mu_t^{(n)} \rangle$

$\mu_t^{(n)} \xrightarrow{w} \mu_t$. And $L_{a,b,\mu_t^{(n)}} \varphi(t) \to \square$ in $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ from A_2'), A_3').

So: $\langle \square^{(n)}, \mu_t^{(n)} \rangle \longrightarrow \langle \square, \mu_t \rangle$

Apply DCT and A_1') again. We obtain the conclusion.

Rmk: Under suitable conds on a_{ij}, b_i. It can also be applied in Nemytskii case.