Nonlinear MPs

We have establish DDSDE $\rightleftarrows$ NLFPE before

But not yet: DDSDE $\leftrightarrow$ Markov Process $\leftrightarrow$ NLFPE

Even for well-posed DDSDE. doesn't satisfy Markov property.

<u>Rmk</u>: Its solution satisfies flow property (by corresp to NLFPE). but it's only for single law. While Markov property is prop. for family of path laws.

(1) <u>Definition</u>:

$\Omega_s := C([s,\infty),\mathbb{R}^d)$ with Zlu. $\Pi_r^s : \begin{array}{l} \Omega_s \to \Omega_r \\ w \mapsto w|_{[r,\infty)} \end{array}$

<u>Def</u>: $\mathcal{P}_0 \subset \mathcal{P}$. Nonlinear MP is family $(\mathbb{P}_{s,\zeta})_{\mathbb{R}_+\times\mathcal{P}_0}$ s.t. $\mathbb{P}_{s,\zeta}$ is p.m. on $\mathcal{B}_{\Omega_s}$ and

i) $\mu_t^{s,\zeta} := \mathbb{P}_{s,\zeta}\circ(\pi_t^s)^{-1} \in \mathcal{P}_0$. $\forall 0\le s\le t$, $\zeta\in\mathcal{P}_0$.

ii) (Nonlinear Markov property): $A \subset \mathcal{B}_{\mathbb{R}^d}$

$\mathbb{P}_{s,\zeta}(\pi_t^s \in A \mid \mathcal{F}_{s,r})(w) = \mathbb{P}_{(s,\zeta),(r,\pi_r^s(w))}(\pi_t^r \in A)$. $\mathbb{P}_{s,\zeta}$-a.s. $\forall 0\le s\le r\le t$, $\zeta\in\mathcal{P}_0$.

where $P_{(s,\zeta),(r,Z^s_r(\omega))}(Z^r_t \in A) = \mathbb{P}_{r,\mu^{s,\zeta}_r}(Z^r_t \in A \mid Z^r_r = Z^s_r(\omega))$

Rmk: i) $\mathcal{P}_0$ can be thought as the class of "allowed initial data"

ii) The motivation of NLMP is to corresp. solution $(\mu^{s,\zeta}_t)$ of NLFPE.

Note that $\mathbb{P}_{r,\mu^{s,\zeta}_r} \neq \int \mathbb{P}_{r,y}\, d\mu^{s,\zeta}_r(y)$ which holds in time-inhomo. MP.

Rather the disintergration family can be replaced by $(P_{(s,\zeta),(r,y)})$

iii) Also in Markov property, we generalize it by recording (s,ζ) on index of RMS rather as classical case.

It's consistent with: $(\mu^{s,\zeta}_t)$ is sol. of NLFPE. And fix $(\mu^{s,\zeta}_t)$. Start from another $(r, Z^s_r(\omega))$ of $\mu^{s,\zeta}_t$-LFPE.

But $\mu^{s,\zeta}_{\cdot}$ still be influential.

prop. $P_{(s,\xi),(r,Z_r^s(w))}(C) = Q_w(C)$. $\forall (w, C) \in \Omega_s \times \mathcal{B}_{\geq r}$. where Q_w is r.c.p. of $\mathbb{P}_{s,\xi}$ over Z_r^s restricted on $\sigma(Z_u^s \mid u \geq r)$

Pf. $RHS \overset{a.s.}{=} \mathbb{E}_{\mathbb{P}_{s,\xi}}(I_C \mid Z_r^s)(w)$

$= \mathbb{E}_{\mathbb{P}_{s,\xi}}(\mathbb{E}(I_C \mid \mathcal{F}_{s,r}) \mid Z_r^s)(w) = LHS.$

prop. One-dim marginals $\mu_t^{s,\xi}$ of NLMP satisfies flow property

Pf: $\forall A \in \mathcal{B}_{\mathbb{R}^d}$. $\mu_t^{s,\xi}(A) = \mathbb{E}_{\mathbb{P}_{s,\xi}}(\mathbb{P}_{s,\xi}(Z_t^s \in A \mid \mathcal{F}_{s,r})(w))$

$= \mathbb{E}_{\mathbb{P}_{s,\xi}}(\mathbb{P}_{r,\mu_r^{s,\xi}}(Z_t^r \in A \mid Z_r^s)(w)) = \mu_t^{r,\mu_r^{s,\xi}}(A)$

Rmk: ⅰ) $(\mu_t^{s,\xi})$ doesn't satisfy time-inhomo

C-K equation: $\mu_t^{s,x} = \int \mu_t^{r,y} \, d\mu_r^{s,x}(y)$

ⅱ) $\mathbb{P}_{r,\mu_r^{s,\xi}} \circ (Z_r^r)^{-1} = \mu_r^{r,\mu_r^{s,\xi}} \overset{flow}{=} \mu_r^{s,\xi}$.

$\Rightarrow$ We proved Rmk ⅱ) above!

prop. $(\mathbb{P}_{s,\xi})_{\mathbb{R}_+ \times \mathcal{P}_0}$ is NLMP. For $\xi \in \mathcal{P}_0$. $s \leq r \leq t$.

Set $P_{r,t}^{s,\xi}(x, dz) := P_{(s,\xi),(r,x)} \circ (Z_t^r)^{-1}(dz)$

$d\mu_r^{s,\xi}$ - a.s. Then: $\mathbb{E}_{\mathbb{P}_{s,\xi}}(f(Z_{t_0}^s, \cdots Z_{t_n}^s)) =$

$$\int_{\mathbb{R}^d} (\cdots \int_{\mathbb{R}^d} f(X_0, \cdots X_n) P^{s,\xi}_{t_{n-1}, t_n}(X_{n-1}, dX_n) \cdots) \mu^{s,\xi}_{t_0}(dX_0)$$

<u>Pf</u>: $LHS = E_{s,\xi}(E_{s,\xi}(\cdot \mid \mathcal{F}_{s,t_0}))$

$= E_{s,\xi}(E_{t_0, X_0}(f(X_0, \cdots)))$

$=$ induction $= E_{s,\xi}(E_{t_0,X_0}(\cdots E_{t_k, X_k}(\cdots)))$

<u>Rmk</u>: i) Finite-dim path law of NLMP is uniquely determined by one-dim time marginal $(P^{s,\xi}_{r,t}(x, dz))$

ii) Even if $\mathcal{P}_0 = \mathcal{P}$, $P^{s,\xi}_{r,t}(x, dz) \neq \mathbb{P}_{r,\delta_x} \circ (Z^r_t)^{-1}(dz)$. i.e. the one-dim marginal to determine path law isn't w.r.t. $(\mathbb{P}_{r,\delta_x})$.

<u>Prop.</u> $(\mathbb{P}_{s,x})$ is classical time-inhomo. MP.

$\Rightarrow (\mathbb{P}_{s,\xi})_{\mathbb{R}_+ \times \mathcal{P}}$ is NLMP with $\mathcal{P}_0 = \mathcal{P}$.

where $\mathbb{P}_{s,\xi} := \int \mathbb{P}_{s,x}\, \xi(dx)$.

<u>Pf</u>: Note $\mathbb{P}_{r,\mu^{s,\xi}_r} = \int \mathbb{P}_{r,y}\, d\mu^{s,\xi}_r(dy)$

$\Rightarrow P_{(s,\xi),(r,y)} = \mathbb{P}_{r,y} = \mathbb{P}_r(\cdot \mid Z^r_r = y)$

which's reduced to the classical markov property.

If $(\mathbb{P}_{s,\zeta})_{\mathbb{R}_+\times\mathcal{P}}$ is NLMP, consisting of sol. law to DDSDE $\Rightarrow$ $(\mu_t^{s,\zeta})$ solves corresp'd NLFPE. And the curve $(p^{s,\zeta}_{r,t}(x,dz))_{t\geq r}$ are weakly conti. p.m. solutions to $\mu^{s,\zeta}$-LFPE. with initial (r,δ_x). $\mu^{s,\zeta}_r$-a.e. x.

$\Rightarrow$ if $\forall (s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0$. $\mu^{s,\zeta}$-LFPE has a unique weakly conti. p.m solution for $\forall$ datum (r,δ_x) $\overset{(A)\ \text{linear mp}}{\Longrightarrow}$ $(p^{s,\zeta}_{r,t})$ is trans. kernel of a linear time-inhomo Markov process $\{\mathbb{P}^{s,\zeta}_{r,x}\}_{r\geq s}$

And it's related to NLMP by:

$\mathbb{P}_{r,\mu_r^{s,\zeta}} = \int_{\mathbb{R}^d} \mathbb{P}^{s,\zeta}_{r,x}\, d\mu^{s,\zeta}_r(x)$ $(*)$. ((s,ζ) $(r,z_r^{\zeta})\sim\mu_r^{s,\zeta}$ t)

Rmk: i) So from prop. above, the finite-dim marginal of $\mathbb{P}_{s,\zeta}$ are uniquely determined by the sol's of linear MPs

ii) Reversely, given $\{(\mathbb{P}^{s,\zeta}_{r,x})_{r,x}\}_{s,\zeta}$ family of MPs and let $(*)$ holds. We get $(\mathbb{P}_{s,\zeta})$.

Even let $\mathcal{P}_0=\mathcal{P}$. $\mu^{s,\zeta}_t := \mathbb{P}_{s,\zeta}\circ(Z^{\zeta}_t)^{-1}$

DDSDE (s, ζ)

Fix $\downarrow \mu^{s,\zeta}$

$\mu^{s,\zeta}$ - SDE

$\downarrow$

sol. $(\mathbb{P}^{s,\zeta}_{r,y})$

$\S$

family of classical mps

can't be solution for a NLFPE.

This is because the argument above holds for $\forall \mu$ satisfies cond. (D).

μ needn't be sol. of a NLFPE

(2) <u>Construction of NLmps</u>:

Next, we don't impose any regularity on coefficient. below also works for Nemytskii's

<u>Denote</u>: i) $\mathcal{M}^{s,\zeta} := \{$ weakly conti. p.m. sol. to NLFPE from $(s,\zeta) \in \mathbb{R}^+ \times \mathcal{P}$, s.t. $a, b \in L^1([0,T] \times \mathbb{R}^d, d\mu_t dt)\}$.

ii) $\mathcal{M}^{s,\zeta}_\eta := \{$ replace NLFPE by η-LFPE on above $\}$.

iii) $\mathcal{M}^{s,\zeta}_{\eta,ex} := \{\mu \in \mathcal{M}^{s,\zeta}_\eta \mid \mu$ is extreme pt. i.e. if $\mu = \alpha\mu^1 + (1-\alpha)\mu^2$. for some $\alpha \in (0,1)$. $\mu^i \in \mathcal{M}^{s,\zeta}_\eta \Rightarrow \mu^1 = \mu^2\}$.

Thm: $\{\mu^{s,\zeta}\}_{\mathbb{R}^+\times\mathcal{P}_0} \subset \mathcal{P}_0 \subseteq \mathcal{P}$ is sol. flow to NLFPE s.t. $\mu^{s,\zeta} \in \mathcal{M}^{s,\zeta}_{\mu^{s,\zeta},ex}$ $\forall (s,\zeta)$.

Then: i) $\forall (s,\zeta)$. $\exists$ unique weak sol. $X^{s,\zeta}$ to the DDSDE with datum (s,ζ) and one-dim marginals $= (\mu^{s,\zeta})$.

ii) $\mathbb{P}_{s,\zeta} := \mathcal{L}X^{s,\zeta}$ is a NLMP. And its one-dim marginal are $\mu^{s,\zeta}_t$.

NLFPE

$\downarrow$

NLMP

$\int$

DDSDE

Rmk: i) Assert i) means the subclass of weak solutions has singleton. rather than the solution is unique and has extra property

ii) If NLFPE is well-posed. Then: $|\mathcal{M}^{s,\zeta}_{\square,ex}| = 1$. And sol. has flow prop. So satisfy cond. above.

Pf: Set $A_{s,\le}(\mu, c) = \{(\eta_t)_{t\ge s} \in C([s,\infty), \mathcal{P})$ $\eta_t \le C\mu_t,\ \forall t \ge s\}$. $A_{s,\le} = \bigcup_{c\ge 1} A_{s,\le}(\mu, c)$.

Rmk: $c \ge 1$ is necessary since $\eta_t \in \mathcal{P}$.

Lem 1: $(s,\xi) \in \mathbb{R}_{\geq 0} \times \mathcal{P}$, $\eta \in C([s,\infty), \mathcal{P})$ and $(\mu_t)_{t \geq s} \in \mathcal{M}_\eta^{s,\xi}$. Then:

$$\mathcal{M}_\eta^{s,\xi} \cap A_{s,\xi}(\mu) = \{\mu\} \iff \mu \in \mathcal{M}_{\eta,ex}^{s,\xi}.$$

Rmk: i) To use it, we let $\eta = \mu^{s,\xi}$

ii) It holds for $\forall$ linear FPE by ignoring "η".

Pf: $\mu \in \mathcal{M}_\eta^{s,\xi} \cap A_{s,\xi}(\mu)$, is obvious

$(\Rightarrow)$ Otherwise $\mu = t\mu^1 + (1-t)\mu^2$

$\mu^1 \neq \mu^2$, $t \in (0,1)$. So:

$\mu^i \in \mathcal{M}_\eta^{s,\xi} \cap A_{s,\xi}(\mu)$, $i = 1,2$.

$(\Leftarrow)$ For $\nu \in \mathcal{M}_\eta^{s,\xi} \cap A_{s,\xi}(\mu)$, $\exists\, \ell_t$

s.t. $\nu_t = \ell_t \mu_t$, $\ell_t \leq c$, $c > 1$

$$\mu_t = \frac{1}{c}\ell_t \mu_t + (1 - \frac{1}{c}\ell_t)\mu_t$$

$$= \frac{1}{c}\nu_t + (1-\frac{1}{c})\lambda_t.$$

where $\lambda_t = (1 - \frac{1}{c}\ell_t \big/ {1-\frac{1}{c}}) \cdot \mu_t$

$\Rightarrow \lambda_t = \nu_t \Rightarrow \nu_t = \mu_t$.

Lem 2: For linear FPE with datum $(s_0, \xi_0) \in \mathbb{R}^+ \times \mathcal{P}$.

$\Rightarrow$ i) If the solution of it is unique

Then: $\forall \eta \sim \zeta_0$. (s_0, η)-s.l. is unique

ii) If $\nu_t^{s_0, \zeta_0}$ is unique sol. in $A_{s_0, \varepsilon}(\nu^{s_0, \zeta_0})$

Then in $A_{s_0, \varepsilon}(\nu^{s_0, \zeta_0})$. $\forall (s_0, g\zeta_0)$-solution is unique. for $g \in \mathcal{B}_b^+(\mathbb{R}^d)$.

$\int g(x) \zeta_0(dx) = 1$. $g \geq \text{const} > 0$.

Lem.[3] $0 \leq s \leq r$. $\mathbb{P} \in \mathcal{P}(\Omega_s)$ is sol. to a linear mp start at s. $\ell \in \mathcal{G}_{s,r}$. bdd : $\Omega_s \to \mathbb{R}^1$. prob. density. w.r.t $\mathbb{P}$. Then:

$(\ell \mathbb{P}) \circ (\pi_r^s)^{-1}$ solves same mp. where

$\pi_r^s : (W_u)_{u \geq s} \mapsto (W_u)_{u \geq r} \in \Omega_r$.

Pf of i):

By superposition Thm. weak sol. $X^{s,\zeta}$ to the DDSDE exists for $\forall (s, \zeta)$ datum.

By Lem[1]: $(\mu_t^{s,\zeta})_{t \geq r}$ is the unique sol. of $\mu^{s,\zeta}$-LFPE. from $(r, \mu_r^{s,\zeta})$ in $A_{r,\varepsilon}(\mu^{s,\zeta})$.

Next, we prove: $\forall (s, \xi) \in \mathbb{R}^+ \times \mathcal{P}_0$, $r \geq s$. Any μ-lmp with 1-dim marginals $\mu_t^{s,\xi}$ has unique sol. from $(r, \mu_r^{s,\xi})$ for $\mu \in A_{s,\xi}(\mu^{s,\xi})$.

(Then: corresp'd SDE has such sol. uniquely. If DDSDE has 2 sol. as in i). Then: $\mu^{s,\xi}$-lmp has 2 sol. contradice!)

<u>Pf</u>: If $\mathbb{P}^i$ are 2 such sol. $i = 1, 2$.

$\Rightarrow \mathbb{P}_t^i := \mathbb{P}^i \circ (Z_t^r)^{-1} = \mu_t^{s,\xi}$.

By π-λ Thm: Check $\mathbb{E}_{\mathbb{P}^1}(H_n) = \mathbb{E}_{\mathbb{P}^2}(H_n)$

for $\forall H_n = \prod_i^n h_i(Z_{t_i}^r)$. $h_i \in \mathcal{B}_b$. $h_i \geq c_i > 0$

We induct on n:

Set $\ell = H_n / \mathbb{E}_{\mathbb{P}^1}(H_n)$. $\ell \in (\frac{1}{c}, c)$ for some $c > 1$. And $\mathbb{E}_{\mathbb{P}^i}(H_n) = 1$ by induction hypothesis.

And $\int_{\mathbb{W}} f(Z_{t_n}^r)\, (\ell\, d\mathbb{P}^1) = \int \square\, \ell\, d\mathbb{P}^2$. $\forall f \in \mathcal{B}_b^{\geq 0}$ from induct hypothesis

(i.e. $\tilde{H}_n = \prod_i^{n-1} h_i(Z_{t_i}^r) \cdot (h_n(Z_{t_n}^r) f(Z_{t_n}^r))$)

$\Rightarrow (\mathcal{L}\mathbb{P}^1)\circ(\pi^r_{t_n})^{-1} = (\mathcal{L}\mathbb{P}^2)\circ(\pi^r_{t_n})^{-1}$

With Lem 3: $(\mathcal{L}\mathbb{P}^i)\circ(\pi^r_{t_n})^{-1}$ solves same LMP from t_n with same datum.

So $\eta^i := (\mathcal{L}\mathbb{P}^i)\circ(\pi^r_{t_n})^{-1}$ both sol. $\mu^{s,\zeta}$-LFPE. We can also check: $\eta^i_t \sim \mu^{s,\zeta}_t$ for $\forall t \geq t_n$.

By Lem 1: $(\eta^1_t)_{t \geq t_n} = (\eta^2_t)_{t \geq t_n}$.

$\Rightarrow \mathbb{E}_{\mathbb{P}^i}(H_n \cdot h_{n+1}(\pi^r_{t_{n+1}})) / \mathbb{E}_{\mathbb{P}^i}(H_n)$

$= \int h_{n+1}(x)\, \eta^i_{t_{n+1}}(dx)$.

So: $\mathbb{E}_{\mathbb{P}^1}(H_{n+1}) = \mathbb{E}_{\mathbb{P}^2}(H_{n+1})$.

<u>Pf of ii):</u>

Note in i). $\mathbb{P}_{s,\zeta}$ p.m. of $X^{s,\zeta}$ has 1-dim marginal $(\mu^{s,\zeta}_t)_t$ correspd the DDSDE.

Next, we need to check $\mathbb{P}_{s,\zeta}$ has nonlinear Markov property.

By 2-λ Thm, we prove: for $\forall h \in \mathcal{B}_b^{\geq 0}$. s.t.

$\exists\, a>1.\ \frac{1}{a}<h<a.\ \forall s\le t_1\le\cdots\le t_n\le r\le t.$ We have:

$\mathbb{E}_{\mathbb{P}^{s,y}}(h(Z^s_{t_1},\cdots Z^s_{t_n})\, I_{Z^s_t}(A)) =$

$\int_{\Omega_s} P_{(s,y),(r,Z^s_r(w))}(Z^r_t\in A)\, h(Z^s_{t_1},\cdots Z^s_{t_n})\,\mathbb{P}_{s,y}(dw)$

where $P_{(s,y)(r,y)}$ is disintegration of $\mathbb{P}_{r,\mu^{s,y}_r}$

w.r.t Z^r_r: $\mathbb{P}_{r,\mu^{s,y}_r}(\cdot)=\int P_{(s,y)(r,y)}(\cdot)\,d\mu^{s,y}_r(dy)$.

$(\Rightarrow h=I_B.\ B\in\mathcal{G}_{s,r}.\ LHS=\mathbb{E}_{\mathbb{P}^{s,y}}(I_B\,\mathbb{P}_{s,y}$

$(Z^s_t\in A\,|\,\mathcal{G}_{s,r}))=RHS=\mathbb{E}_{\mathbb{P}^{s,y}}(P_\square(\square)\,I_B).)$

First note that $P_{(s,y),(r,y)}$ solves $\mu^{s,y}$-emp

with datum (r,δ_y). by last remark in (1)

For $\forall e\in\mathcal{B}_b^{\ge 0}$. s.t. $\int e\,d\mu^{s,y}_r=1$.

$\Rightarrow \mathbb{P}_e:=\int_{\mathbb{R}^n}P_{(s,y),(r,y)}(e(y)\,d\mu^{s,y}_r(y))$ Solves emp

with datum $(r,e\mu^{s,y}_r)$ (As a Lem.)

Set $\mathbb{E}_{\mathbb{P}_{s,y}}(h(Z^s_{t_1}\cdots Z^s_{t_n})\,|\,\sigma(Z^s_r))=\tilde{g}(Z^s_r)$

$g=C_0\tilde{g}$. s.t. $\int g\,d\mu^{s,y}_r=1$. Let $e=g$

Also. $\theta:\Omega_s:\to\mathbb{R}^1$. $\theta\overset{\Delta}{=}C\,h(Z^s_{t_1}\cdots Z^s_{t_n})$

So $\mathbb{E}_{\mathbb{P}^{s,y}}(\theta)=1$. Let $\mathbb{P}^\theta:=(\theta\,\mathbb{P}_{s,y})\cdot(Z^s_r)^{-1}$

Apply Lem[3] again. $\Rightarrow \mathbb{P}^{\theta}$ solves same emp. with $(r, g\mu_r^{s,s})$. (check $\mathbb{P}^{\theta}\circ(Z^r_r)^{-1}(A) = \square$)

Besides, $\mathbb{P}^{\theta}\circ(Z^r_t)^{-1}$, $\mathbb{P}_g\circ(Z^r_t)^{-1} \ll \mu_t^{s,s}$

So belong to $\mathcal{A}_{r,s}(\mu^{s,s})$.

$\Rightarrow \mathbb{P}^{\theta}\circ(Z^r_t)^{-1} = \mathbb{P}_g\circ(Z^r_t)^{-1}$

Now LHS of the eq. we want to prove

$= C_0^{-1}\,\mathbb{P}^{\theta}\circ(Z^r_t)^{-1}(A) = C_0^{-1}\,\mathbb{P}_g\circ(Z^r_t)^{-1}(A) = \cdots = \text{RHS}$

Cor. $\mathcal{B}_0 \subset \mathcal{P}_0 \subset \mathcal{P}$. $(\mu_t^{s,s})$ is sol. flow to NLFPE. st. $\mu_t^{s,s} \in \mathcal{B}_0$, $\forall t \geq s$. if $s \in \mathcal{B}_0$ & $\mu_t^{s,s} \in \mathcal{B}_0$, $\forall t > s$ if $s \in \mathcal{P}_0$ (trapped)

If $(\mu^{s,s}) \in \mathcal{M}^{s,s}_{\mu^{s,s},ex}$. $\forall (s,s) \in \mathbb{R}^{\geq 0}\times\mathcal{B}_0$

Then $\exists$ nonlinear Mp $(\mathbb{P}_{s,s})_{\mathbb{R}^{\geq 0}\times\mathcal{P}_0}$ st. $\mathbb{P}_{s,s}\circ(Z^r_t)^{-1} = \mu_t^{s,s}$ $\forall (s,s) \in \mathbb{R}^{\geq 0}\times\mathcal{P}_0$. Consists of corresp. path law of weak sol. for DDSDE.

Moreover, if $s \in \mathcal{B}_0 \Rightarrow \mathbb{P}_{s,s}$ is the unique weak s.l. to the DDSDE with marginals $(\mu_t^{s,\zeta})$.

Rmk: Set $\mathcal{B}_0 = \mathcal{P} \cap L^\infty$. $\mathcal{P}_0 = \mathcal{P}_1 \Rightarrow \exists$ possible sol. flow $(\mu^{s,\zeta})_{\mathbb{R}_+ \times \mathcal{P}}$ to NLFPE will $\in \bigcap_{\delta > s} L^\infty((\delta, \infty), L^\infty)$.

It's called $L^1 - L^\infty$ regularization

(Note $\mu_t^{s,\zeta}$ will be regular as it evolves. think $\delta_x \in L^1$ but not in L^∞)

(3) Applications:

① Examples:

i) Well-posed NLFPE:

If it has unique weakly conti. sol $\mu^{s,\zeta}$ & $\mu^{s,\zeta}$ - LFPE also well-posed. Then, by main Thm in (2). We have unique NL Markov process with marginal $= \mu$

Rmk: But these well-poseness can't hold for Nemytskii type coefficients.

ii) Generalized PME:

$\partial_t u(t,x) = \Delta\beta(u) - \mathrm{Div}(D(x)B(u(t))u(t))$

For β, D, B regular enough. $\mathcal{P}_0 := \mathcal{P} \cap L^\infty$

$\exists$ sol. $\mu_t^{s,\zeta} = u_t^{s,\zeta}(x)dx$ is weakly conti. in

$\in \bigcap_{s<T} L^\infty((s,T)\times\mathbb{R}^d)$ and has flow prop in $\mathcal{P}_0$. And it's unique in $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$

which solves $\mu^{s\zeta}$ - LFPE from (s,ζ).

So: Since $\mathcal{A}_{s,\zeta}(\mu) \le \bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$

$\Rightarrow$ Apply main Thm in (2) admits $\mathbb{P}_{s,\zeta}$ which is path-law for DDSDE:

$dX_t = B(u_t(X_t))D(X_t)dt + \sqrt{\frac{2\beta(u_t(X_t))}{u_t(X_t)}}\,dB_t$

$\mathcal{L}_{X_t} = u_t dx$, $t \ge s$.

Rmk: D Drift term is easy to obtain.

Note $\Delta\beta(u) = \Delta(\beta(u)/u \cdot u)$.

So, we artifically set $\frac{1}{2}\sigma\sigma^T$

$= \beta(u)/u$. $\Rightarrow$ Obtain σ term.

ii) The PDE sol. u has probabilitic

repre. as marginal of MLMP.

iii) Classical PME:

$\partial_t u = \Delta(|u|^{m-1}u)$. $m \geq 1$.

$\forall (s,\zeta) \in \mathcal{P}_0 = \mathcal{P}$. $\exists$ unique weakly conti.

p.m. $u^{s,\zeta}$ in $\bigcap_{T>t>s} L^\infty([t,T]\times\mathbb{R}^d) \subset A_{s,\zeta}(\mu)$

So it has flow prop. from uniqueness.

And for $\zeta = \delta_x$. u^{s,δ_x} can be written

explicitly (called Barenblatt sol.)

It corresponds to DDSDE.

$dX_t = (2u_t(X_t)^{m-1})^{\frac{1}{2}}dB_t$. $\mathcal{L}_{X_t} = u_t(x)dx$.

Set $\mathcal{P}_0 = \mathcal{P}$. $\mathcal{B}_0 = \mathcal{P}\cap L^\infty$.

If coeff. is regular again. results in

example ii) also works here $\Rightarrow$ Apply cor.

with $(\mathcal{B}_0, \mathcal{P}_0)$ in (2).

② p-Brownian Motions:

Consider p-Laplace equation where $p>2$:

$$\partial_t u(t,x) = \mathrm{Div}(|\nabla u|^{p-2}\nabla u), \quad (t,x)\in\mathbb{R}^{>0}\times\mathbb{R}^d.$$

Rmk: The sol. can be given explicitly:

$$w^y(t,x) = t^{-k}\left(C - q\,t^{-kp/d(p-1)}|x-y|^{\frac{p}{p-1}}\right)_+^{p-1/p-2}.$$

called Barenblatt sol.

Also $w^y \xrightarrow{w} \delta_y$ as $t\to 0$.

Note that it doesn't Fokker-Planck type

Prop. (FP Reformulation)

$t\mapsto w^y(t,x)\,dx$ is weakly conti. p.m.

sol. with datum δ_y to NLFPE:

$$\partial_t u = \Delta(|\nabla u|^{p-2}\nabla u) - \mathrm{Div}(\nabla(|\nabla u|^{p-2})u)$$

Rmk: i) It correspd to DDSDE:

$$dX_t = \nabla(|\nabla u(t,X_t)|^{p-2})\,dt + \sqrt{2}\,|\nabla u(t,X_t)|^{\frac{p-2}{2}}\,dW_t, \quad \mathcal{L}_{X_t} = u(t,x)dx.$$

ii) Note that the NLFPE coeff.

depend on nbd of u rather than single value from ∇u.

So the NL superposition prin. doesn't work here.

<u>Prop.</u> $\exists$ weak sol. $(X_t^y)_{t\geq 0}$ to DDSDE in Rmk i) above. St. $\mathcal{L}_{X_t^y}(dx) = W^y(t,x)dx$ for $t>0$ & $\mathcal{L}_{X_0^y}(dx) = \delta_y(dx)$.

<u>Pf.</u> Apply linear superposition princi. by frozening W^y (i.e. ∇W^y) $\Rightarrow \partial_t u = \Delta(|\nabla W^y|^{p-2}u) - \mathrm{Div}(\nabla(|\nabla W^y|^{p-2}u))$

<u>Def.</u> i) $\mathcal{P}_0 := \{W^y(\delta,x)dx,\ y\in\mathbb{R}^d,\ \delta\geq 0\} \subset \mathcal{P}$. i.e. all possible distr. from Barenblatt sol.

<u>Rmk:</u> i) $\{\delta_y\}_{\mathbb{R}^d} \subset \mathcal{P}_0$.

ii) $\forall \zeta \in \mathcal{P}_0$. $\exists$ unique (δ,y). st $\zeta = W^y(\delta,x)dx$.

ii) For $\zeta = W^y(\delta,x)dx \in \mathcal{P}_0$. We set $\mu_t^{s,\zeta} := W^y(\delta+t-s,x)dx$. $\forall t\geq s$.

Rmk: $(\mu_t^{s,\zeta})_{t\geq s}$ is weakly conti. sol. to NLFPE of W^γ with datum (s,ζ). And by construction. $\Rightarrow$ $(\mu_t^{s,\zeta})$ is sol. flow.

Thm$^{(*)}$. For $d\geq 2$, $p>2$. $\exists$ NLMP $(\mathbb{P}_{s,\zeta})_{s\geq 0,\zeta\in\mathcal{P}_0}$ s.t. it has marginal $(\mu_t^{s,\zeta})$. And it's path law of $X_t^{s,\zeta}$ to DDSDE of W^γ. Besides. $X^{s,\zeta}$ is the unique weak sol with marginal $\mu_t^{s,\zeta}$ to the DDSDE.

Rmk: $(\mathbb{P}_{s,\zeta})$ is uniquely determined by the DDSDE & Barenblatt sol.

Lem. $(\mathbb{P}_{s,\zeta})$ above is time-homo. i.e. $\mathbb{P}_{s,\zeta} = \mathbb{P}_{0,\zeta}\circ(\hat{\Pi}_s)^{-1}$. $\forall (s,\zeta)$. $\hat{\Pi}_s((W_t)_{t\geq 0}) = (W(t-s))_{t\geq s}$. Besides. for $\zeta = W^\gamma(s,x)dx$. $\Rightarrow \mathbb{P}_{0,\zeta} = \mathbb{P}_{0,\zeta}\circ(\tilde{\Pi}_\zeta)^{-1}$. where $\tilde{\Pi}^r((W_t)_{t\geq s}) = (W(t+r))_{t\geq s}$.

Rmk: i) $\mathbb{P}_{s,\gamma}$ is uniquely determined by $\{\mathbb{P}_y := \mathbb{P}_{0,\delta_y}\}$.

ii) For translation $T_y : \begin{array}{c} \mathbb{C}_{\geq 0} \to \mathbb{C}_{\geq 0} \\ w \mapsto w+y \end{array}$

$\mathbb{P}_y \neq \mathbb{P}_0 \circ T_y^{-1}$. except $p=2$.

Def: For $d \geq 2$, $p > 2$. $\{\mathbb{P}_y\}$ defined above is called p-BM, and $\mathbb{P}_0$ is p-Wiener measure.

Rmk: For $p=2$, the p-Laplace eq. is just heat eq. So it corresponds to BM case.

Restricted Uniqueness:

Note if we let $\mathcal{B}_\gamma := \{W^\gamma(\delta, x)dx\}_{\delta > 0}$

$\forall (s,\gamma) \in [0,\infty) \times \mathcal{B}_\gamma$. Thm$^{(*)}$ can be proven by applying Cor. in (2)

Besides. $\partial_t u = \Delta(|\nabla W^\gamma(\delta+t-s,x)|^{p-2} u) - \nabla(|\nabla W|^{p-2} u)$

$u(t,x)dx \xrightarrow{t \to s} \gamma$ has a unique sol. $(u_t)_{t \geq s}$.

under restrict: $\exists c > 0$. s.t. $0 \leq u \leq c\, W^\gamma(\delta+t-s, x)$