(2) General principle:

Consider X is polish space. $I: X \to \mathbb{R}^{\geq 0}$ is a proper rate func. i.e. $K_L = \{ I(x) \leq L \}$ is cpt for $\forall L$. and I is l.s.c.

Def: For (P_n) seq of laws on X. it satisfy large deviation principle (LDP) on X at speed n. w.r.t. $I(\cdot)$ if:

i) $\overline{\lim_{n\to\infty}} \frac{1}{n} \log P_n(C) \leq - \inf_C I(x)$. $\forall C \underset{\text{close}}{\subset} X$.

ii) $\underline{\lim_{n\to\infty}} \frac{1}{n} \log P_n(O) \geq - \inf_O I(x)$. $\forall O \underset{\text{open}}{\subset} X$

Rmk: i) Note for i.i.d seq case.

it satisfies LDP.

ii) Note $P_n(X) = 1 \Rightarrow \inf I(x)$ must be 0. for (P_n)

Thm: For (P_n) satisfies LDP at speed n. w.r.t $I(\cdot)$. on X. Then. $\forall L < \infty$. $\exists D^L \underset{\text{cpt}}{\subset} X$. st. $P_n(D^L) \geq 1 - e^{-nL}$. $\forall n$.

Pf: Consider $A^L = \{ I(x) \leq L+2 \}$. cpt

Suppose A_k is finite union of open balls with $r = 1/k$ cover A^L.

By LDP : $\overline{\lim_{n\to\infty}} \frac{1}{n} \log P_n(A_k^c) \le -(l+2)$ ($I \ge l+2$ on A_k^c)

$\Rightarrow \exists\, n_0 = n_0(k).\ \forall n \ge n_0 : P_n(A_k^c) \le e^{-n(l+1)}$

WLOG. set $n_0(k) \ge k$. $\forall k \ge 1$.

$\Rightarrow P_n(A_k^c) \le e^{-nl} \cdot e^{-k}$.

On the other hand. For each $k \ge 1$.

$\exists\, (B_{k,i})_{i \le n_0(k)}$ seq of cpt sets. s.t.

$P_i(B_{k,i}^c) \le e^{-k} \cdot e^{-il}$. $\forall\, i \le n_0(k)$.

Let $D^l = \bigcap_k (\bar{A}_k \cup (\bigcup_{j=1}^{n_0(k)} B_{k,j}))$

$P_n((D^l)^c) \le \sum_{k\ge 1} P_n(A_k^c \cap (\cap B_{k,j}^c)) \le e^{-nl} \sum e^{-k}$

Since $P_n(A_k^c \cap (\cap B_{k,j}^c)) \le \begin{cases} P_n(A_k^c). & n \ge n_0(k) \\ P_n(B_{k,n}^c) & n \le n_0(k) \end{cases}$

<u>Thm</u> (Product of LDP)

If (P_n). (Q_n) are two seq of laws satisfy LDP w.r.t. $I(\cdot)$ and $J(\cdot)$ on X and Y. Then $R_n =: P_n \times Q_n$ satisfies LDP w.r.t $K(x,y) =: I(x) + J(y)$. on $X \times Y$.

<u>Pf:</u> 1) Consider $z = (x,y) \in X \times Y$.

By l.s.c. of I. $\exists\, U_0$ nbd of x.

s.t. $I(x') \ge I(x) - \varepsilon$. $\forall x' \in U_0$

By separable. $\exists\, U_x$ open. $U_x \subset \bar{U}_x \subset U_0$.

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{1}{n} \log P_n(U_x) \le -I(x) + \varepsilon$, Similarly. $\exists V_y$. of y. Set $N_z = U_x \times V_y$.

$$\Rightarrow \varlimsup_{n\to\infty} \frac{1}{n} \log R_n(N_z) \le -K(z) + 2\varepsilon.$$

2) For $D \subset X \times Y$. cpt set. cover D by finite nbd's N_z (as above).

$$R_n\left(\bigcup_1^N N_z\right) \le N \max_{1\le j\le N} R_n(N_z)$$

Note N will disappear. after taking logarithm and divide n. set $n\to\infty$

$$\Rightarrow \varlimsup_{n\to\infty} \frac{1}{n} \log R_n\left(\bigcup_1^N N_z\right) \le -\inf_D K(z) + \varepsilon.$$

3) For general $C \subset X \times Y$.

Note $\exists A^\ell \underset{cpt}{\subset} X$. $B^\ell \underset{cpt}{\subset} Y$. s.t.

$P_n(A^\ell)^c \le e^{-n\ell}$. $Q_n(B^\ell)^c \le e^{-n\ell}$.

Set $C^\ell = A^\ell \times B^\ell$. $C = C\cap C^\ell + C \cap C^{\ell c}$.

$$\Rightarrow \varlimsup \frac{1}{n} \log R_n(C) \le \max(-\inf_{C^\ell} K(z), -\ell)$$

$$\le \max\{-\inf_C K(z), -\ell\} \xrightarrow[\ell\to\infty]{} -\inf_C K(z)$$

4) For lower bound. since it's local. find $U_x \times V_y =: N$. nbd of $z=(x,y)$. By LDP of P_n. Q_n. we have:

$$\varliminf_{n\to\infty} \frac{1}{n} \log R_n(N) \ge -K(z) \quad \forall z \in O.$$

Thm: (Contraction principle)

$F: X \to Y$. is conti. between. polish spaces.

If (P_n) satisfies LDP with $I(\cdot)$ on X.

Then. $Q_n =: P_n \circ F^{-1}$ satisfies LDP on Y

w.r.t. $J(y) = \inf_{x \in F^{-1}(y)} I(x)$.

Pf: Note F^{-1} retains open/close. easy to check.

Thm (Varadhan's Thm)

If (P_n) satisfies LDP on X w.r.t. $I(\cdot)$.

$F \in C_B^{\mathbb{R}'}(X)$. Then $a_n =: \int_X e^{nF(x)} dP_n(x)$.

satisfies $\lim_{n\to\infty} \frac{1}{n} \log a_n = \sup_x (F(x) - I(x))$.

Rmk: It's some kind of Laplace transf.

Pf: Procedure: nbd of point $\Rightarrow$ cpt set $\Rightarrow$ general.

i) $\forall x \in X$. $\exists N_x$. nbd of x. s.t.

$|F(x') - F(x)| \leq \varepsilon$. $I(x') + \varepsilon \geq I(x)$. $\forall x' \in N_x$.

Set $a_n(A) = \int_A e^{nF(x)} dP_n(x)$

Note $a_n(N_x) \leq e^{n(F(x)+\varepsilon)} P_n(\overline{N_x})$

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{1}{n} \log a_n(N_x) \leq (F(x) - I(x)) + 2\varepsilon$.

2) $\forall D$ cpt in X. similar as before

$\overline{\lim_{n\to\infty}} \frac{1}{n} \log a_n(D) \leq \sup_x (F(x) - I(x)) + 2\varepsilon$.

3') Find (K^L). s.t. $P_n(K^L)^c \le e^{-nL}$

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{1}{n} \log \lambda_n(K^L)^c \le \|f\|_\infty - L$.

So $\lim_{n\to\infty} \frac{1}{n} \log \lambda_n \le \max\{\|f\|_\infty - L, \sup(F(x)-I(x)) + 2\varepsilon\}$

$\xrightarrow[\varepsilon\to 0]{L\to\infty} \sup(F(x)-I(x))$

4') For lower bound. $\forall x \in X$. $\exists N_x$ nbd of x.

$\lambda_n \ge \lambda_n(N_x) \ge e^{n(F(x)-\varepsilon)} P_n(N_x)$

$\Rightarrow \underline{\lim_{n\to\infty}} \frac{1}{n} \log \lambda_n \ge \sup_x (F(x)-I(x)) - \varepsilon$.

__Rmk:__ Note bdd and u.s.c. are for upper bound. While l.s.c. is for lower.

__Def:__ $Q_n(A) =: \dfrac{\int_A e^{nF(x)} dP_n(x)}{\int_X e^{nF(x)} dP_n(x)}$. for $A \in \mathcal{B}_X$.

__Thm.__ For (P_n) satisfies LDP with rate function $I(\cdot)$ and $F \in C_B(X)$. Then (Q_n) also satisfies LDP on X. w.r.t $J(x) =: \sup_x(F-I) - (F(x)-I(x))$

__Pf:__ Note that $Q_n(C) = \lambda_n(C) / \lambda_n$.

argue as before. $\Rightarrow \overline{\lim} \frac{1}{n} \log Q_n(C) \le -\inf_C J$.