Cramer & Schilder

Let E is polish space. $\{\mu_\varepsilon\}_{\varepsilon>0}$ is p.m. on E. s.t. $\mu_\varepsilon \xrightarrow{w} \delta_p$. For $p \in U$. then: $\mu_\varepsilon(U^c) \to 0$. We'd like to evaluate such "deviant" behavior (decay rate)

(1) <u>Cramer Thm:</u>

First assume $\mu_\varepsilon \ll dx$. $\forall \varepsilon$. Since $\mu_\varepsilon \xrightarrow{w} \delta_p$. We ansatz: $\frac{d\mu_\varepsilon}{dx} = g_\varepsilon e^{-I/\varepsilon}$. where g_ε is const. s.t. $\varepsilon \log g_\varepsilon \to 0$. $I(x) \geq 0$ and vanishing at point p. Let Γ. s.t. $dx(\Gamma) < \infty$

By def: $\varepsilon \log(\mu_\varepsilon(\Gamma)) = \log\left(\int_\Gamma g_\varepsilon e^{-I/\varepsilon} dx\right)^\varepsilon$

$$= o(1) + \log\left(\int_\Gamma e^{-I/\varepsilon} dx\right)^\varepsilon$$

Also: $\left(\int_\Gamma e^{-I/\varepsilon} dx\right)^\varepsilon \xrightarrow{\varepsilon \to 0} \operatorname*{ess\,sup}_{z \in \Gamma} e^{-I(z)}$.

$\left(\text{it} \geq \left(\int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{-I/\varepsilon}\right)^\varepsilon \longrightarrow e^{-I(x_0)}. \ \forall x_0 \in \Gamma\right)$

So: $\lim_{\varepsilon \to 0} \varepsilon \log(\mu_\varepsilon(\Gamma)) = -\operatorname*{ess\,inf}_\Gamma I(z)$.

e.g. $\nu_\varepsilon(dq) = (2\pi\varepsilon)^{-d/2} e^{-|q|^2/2\varepsilon} dx$.

$\Rightarrow \lim_{\varepsilon\to 0} \varepsilon \log(\nu_\varepsilon(\Gamma)) = -\text{ess inf}\{\frac{|q|^2}{2} \mid q \in \Gamma\}$

Next, we consider i.i.d. case. For μ p.m. on $\mathbb{R}^1$. $\mu^n := \mu * \cdots \mu$ (n times). μ_n is dist. of $\sum_1^n X_i/n$. $X_i \overset{\text{i.i.d.}}{\sim} \mu$.

We see $\mu_n \xrightarrow{w} \delta_p$. $p = \mathbb{E}_\mu(X) = \int_{\mathbb{R}^1} x\,d\mu$.

Let $\mu_\varepsilon = \mu_n$, if $n-1 < 1/\varepsilon \leq n$. Then we see $\{\mu_\varepsilon\}_{\varepsilon>0}$ is candidate of LDP.

Denote: $\Lambda_\mu(\lambda) := \log \mathbb{E}_\mu(e^{\lambda X})$

Rmk: Λ_μ is convex l.s.c. (By Hölder) as $\uparrow$-limit of smooth func.

$\Rightarrow \Lambda_\mu^* := \sup_{\lambda\in\mathbb{R}^1} \{\lambda x - \Lambda_\mu(\lambda)\}$ is l.s.c and convex.

Lem. μ is p.m. on $\mathbb{R}^1$. $\Rightarrow \Lambda_\mu^* \geq 0$. And

i) If $\mathbb{E}_\mu(|X|) < \infty$. Then $\Lambda_\mu^*(p) = 0$ and $\Lambda_\mu^*(p)$ $\uparrow$ on $[p,\infty)$. $\downarrow$ on $(-\infty, p]$.

Besides, $\Lambda_\mu^*(q) = \sup\{\lambda q - \Lambda_\mu(\lambda) : \lambda \geq 0\}$

and $\mu([q,\infty)) \leq e^{-\Lambda_\mu^*(q)}$ for $q \geq p$.

$\Lambda_\mu^*(q) = \sup\{\lambda q - \Lambda_\mu(\lambda) : \lambda \leq 0\}$ and μ

$((-\infty, q]) \leq e^{-\Lambda_\mu^*(q)}$. for $q \leq p$.

ii) If $\exists$ nbd of 0. st. $\Lambda_\mu(\lambda) < \infty$. Then

$\Lambda_\mu^*(x) \to \infty$ as $|x| \to \infty$.

iii) If $\Lambda_\mu(\lambda) < \infty$. for $\forall \lambda \in \mathbb{R}^1$. Then: Λ_μ

$\in C^\infty(\mathbb{R}^1)$ and $\Lambda_\mu^*(x)/|x| \xrightarrow{|x|\to\infty} \infty$

Pf: i) $\Lambda_\mu^*(q) \geq \lambda q - \Lambda_\mu(\lambda)\big|_{\lambda=0} = 0$.

And with Jensen inequi. We have

$\Lambda_\mu(\lambda) \geq \lambda p$, $\forall \lambda \in \mathbb{R}^1$. $\Rightarrow \Lambda_\mu^*(p) \leq 0$.

Since $\Lambda_\mu^*(\lambda)$ is convex. We get

the 2nd claim.

Also: $\lambda(q-p) \geq \lambda q - \Lambda_\mu(\lambda)$. We get

the 3rd claim.

With $\mu([q,\infty)) \leq e^{-(\lambda q - \Lambda_\mu(\lambda))} \leq e^{-\Lambda_\mu^*(q)}$.

We have the last assertation.

ii) $\exists \lambda_0 > 1$. s.t. $\Lambda_\mu^*(x)/|x| \geq \lambda_0 - \frac{\Lambda_\mu(\lambda_0)}{|x|}$

$\Rightarrow \overline{\lim}_{x\to\infty} \Lambda_\mu^*(x)/|x| \geq \lambda_0$.

iii) By DCT. We get the smoothness

And we set $\lambda_0 \to \infty$ on ii).

Lem. (upper bd)

If $\mathbb{E}_\mu(|X|) < \infty$. Then: $\forall F \underset{\text{close}}{\subseteq} \mathbb{R}^1$. We

have: $\overline{\lim}_{n\to\infty} \frac{1}{n} \log \mu_n(F) \leq -\inf_F \Lambda_\mu^*$.

Pf: Note that $\Lambda_n(\lambda) := \Lambda_{\mu_n}(\lambda) = n\Lambda_\mu(\frac{\lambda}{n})$

So: $\Lambda_n^*(\lambda) = n\Lambda_\mu^*(\lambda)$

By Chebyshev inequi.:

$\mu_n((-\infty, q]) \vee \mu_n([q, \infty)) \leq e^{-n\Lambda_\mu^*(q)}$

For $F \subset [p, \infty)$ or $(-\infty, p]$. Since

Λ_μ^* is mono. on $[p, \infty)$ or $(-\infty, p]$

So we have the result.

For $p \in F$. We set $q_+ = \sup\{x \in F.$

$x \geq p\}$. $q_- = \inf\{x \in F. x \leq p\}$.

$\mu_n(F) \leq e^{-n\Lambda_\mu^*(q_+)} + e^{-\Lambda_\mu^*(q_-)} \leq 2e^{-n\inf_F \Lambda}$

Thm (Crámer)

If $\Lambda_\mu(\lambda) < \infty$, $\forall \lambda \in \mathbb{R}^1$. Then: $\forall \Gamma \in \mathcal{B}_{\mathbb{R}^1}$

We have:

$$-\inf_{\text{int}\Gamma} \Lambda_\mu^* \le \varliminf_{n\to\infty} \frac{1}{n} \log \mu_n(\Gamma) \le \varlimsup_{n\to\infty} \frac{1}{n} \log \mu_n(\Gamma)$$

$$\le -\inf_{\bar{\Gamma}} \Lambda_\mu^*(\lambda)$$

Pf: We only need to show the lower bdd since we have Lemma above.

Next, we show $\forall q \in \Gamma$. for $(q-\delta, q+\delta) \subset \text{int}\Gamma$. we have:

$$\varliminf_{n\to\infty} \frac{1}{n} \log \mu_n((q-\delta, q+\delta)) \ge -\Lambda_\mu^*(q)$$

It's proved by change of measure.

i) If $\exists \lambda \in \mathbb{R}^1$. s.t. $\Lambda_\mu^*(q) = \lambda q - \Lambda_\mu(\lambda)$.

(Assume $\lambda \ge 0$; $\lambda < 0$ is analogous)

Set $\tilde{\mu}(dx) = e^{\lambda x} / e^{\Lambda_\mu(\lambda)} \mu(dx)$. p.m.

We see $\mathbb{E}_{\tilde{\mu}}(X) = \frac{d}{dt} \Lambda_\mu(t) |_{t=\lambda}$

And $q = \frac{d}{dt} \Lambda_\mu(t) |_{t=\lambda}$ by assumpt.

$\Rightarrow \mathbb{E}_{\tilde{\mu}}(X) = q$. So: $\tilde{\mu}_n(-\varepsilon+q, q+\varepsilon) \to 1$

$\mu_n(q-\delta, q+\delta) = \mu^n(\{|\frac{1}{n}\sum_1^n y_k - q| < \delta\})$

$\geq e^{-n\lambda(q+\delta)} \int_{[\cdots]} e^{\lambda\sum_1^n y_k} \mu^n(dy)$

$= e^{-n(\lambda(q+\delta) - \Lambda_\mu(\lambda))} \tilde{\mu}_n(q-\delta, q+\delta) \to \square$

So: $\varliminf_{n\to\infty} \frac{1}{n}\log\mu_n(q-\delta, q+\delta) \geq -\Lambda_\mu^*(q) - \lambda\delta.$

$\Rightarrow \varliminf_{n\to\infty} \frac{1}{n}\log\mu_n(\Gamma) \geq \square \geq -\Lambda_\mu^*(q) - \lambda\delta.$

And then set $\delta \downarrow 0$. We have lower bdd.

ii) If not such λ exists. Then:

$\Lambda_\mu^*(q) > \lambda q - \Lambda_\mu(\lambda)$, $\forall \lambda \in \mathbb{R}^1$.

WLOG. For $q \geq \mathbb{E}_\mu(X)$. Note $\exists(\lambda_k)\uparrow\infty$

(otherwise it attains max.) s.t. $\lambda_k q - \Lambda_\mu(\lambda_k)$

$\uparrow \Lambda_\mu^*(q)$. So: $e^{\Lambda_\mu(\lambda_k) - \lambda_k q} = \int e^{\lambda_k(x-q)} d\mu$

$\to e^{-\Lambda_\mu^*(q)}$. With $\int_{(-\infty,q)} e^{\lambda_k(x-q)} \xrightarrow{DCT} 0$.

So: $\int_{[q,\infty)} e^{\lambda_k(x-q)} d\mu \to e^{-\Lambda_\mu^*(q)}$.

$\Rightarrow \mu((q,\infty)) = 0$. $\mu(\{q\}) = e^{-\Lambda_\mu^*(q)}$.

We have: $\mu_n(\{q\}) = \mathbb{P}(\frac{\sum_1^n Y_i}{n} = q) \geq \mathbb{P}(Y_i = q, \forall i \leq n)$

$= \mu^n((q,\cdots,q)) = e^{-n\Lambda_\mu^*(q)}$.

(2) Schilder's Thm:

For $d \in \mathbb{Z}^+$. Set $\Theta = \{\theta \in C(\mathbb{R}^{\geq 0}, \mathbb{R}^d),\ \theta_0 = 0,$

$\lim_{t\to\infty} |\theta_t|/t = 0\}$ with norm $\|\theta\|_\Theta = \sup_{t\geq 0} \frac{|\theta(t)|}{1+t}$

With $\mathcal{B}_\Theta$ is Borel σ-algebra on Θ.

Rmk: i) Θ is separable Banach space

ii) Θ^* dual space $= \{$ Borel measure λ on $\mathbb{R}^{\geq 0}$. s.t. $\lambda\{0\} = 0$. $\int(1+t)|\lambda| < \infty\}$

with $\|\lambda\|_{\Theta^*} := \int_{\mathbb{R}^{\geq 0}} (1+t)|\lambda|dt < \infty$.

$\langle\lambda, \theta\rangle := \int_0^\infty \theta(t)\cdot\lambda(dt)$.

iii) Set $\mathcal{B}_t := \sigma(ev_s : \theta \in \Theta \mapsto \theta_s,\ s \leq t)$

We have $\mathcal{B}_\Theta = \bigvee_{t\geq 0}^{\infty} \mathcal{B}_t$.

Thm. Wiener measure $W(d\theta)$. $(X_t \sim \mathcal{B}_t)$

has ch.f.: $\mathbb{E}(e^{i\langle\lambda,\theta\rangle}) = e^{-\Delta_W(\lambda)}$.

where $\Delta_W(\lambda) = \int_{(0,\infty)^2} s\wedge t\ \lambda(ds)\lambda(dt)/2$.

for $\lambda \in \Theta^*$.

Rmk: Wiener measure $W(d\theta)$ has a

formal representation: $W(d\theta) = C \cdot \exp(-\frac{1}{2}\int_0^\infty |\dot{\theta}(t)|^2 dt) d\theta$. (Here $C = \infty$. $d\theta$ nonexist.) But let W_ε is distr. of $\theta \mapsto \varepsilon^{\frac{1}{2}}\theta$ under W. We have:

$W_\varepsilon(d\theta) = C_\varepsilon \exp(-\frac{1}{2\varepsilon}\int_0^\infty |\dot{\theta}(t)|^2 dt) d\theta$.

And it's consistent with $W_\varepsilon \xrightarrow{w} \delta_0$.

Next, we get LDP for $\{W_\varepsilon(d\theta)\}_{\varepsilon>0}$.

<u>Lem.</u> Let $\psi_\lambda(t) = \int_0^t \lambda(s,\infty) ds$. $t > 0$. Then:

i) $\psi_\lambda \in \Theta$

ii) $\forall \lambda, \eta \in \Theta^*$. $\int_{(0,\infty)^2} s \wedge t \; \lambda(ds)\, \eta(dt) = \int_{[0,\infty)} \lambda(s,\infty)\, \eta(s,\infty) ds = \langle \eta, \psi_\lambda \rangle$. So:

$\Lambda_W(\lambda) = \frac{1}{2}\int_0^\infty |\lambda(s,\infty)|^2 ds = \frac{1}{2}\langle \lambda, \psi_\lambda \rangle$.

iii) $\Lambda_W^*(\psi) = \begin{cases} \infty & . \text{ if } \psi \notin H^1 \\ \|\psi\|_{H^1}^2/2 & . \text{ if } \psi \in H^1 \end{cases}$

<u>Cor.</u> i) $\Lambda_W^*(\psi_\lambda) = \Lambda_W(\lambda)$. $\forall \lambda \in \Theta^*$

ii) $\forall L > 0$. $\{\psi \in \Theta \mid \Lambda_W^*(\psi) \leq L\} \subseteq_{cpt} \Theta$

$\Rightarrow \Lambda_W^*(\psi)$ is good rate func.

Pf: i) $\varphi_\lambda \in H^1$

ii) Λ_W^* is l.s.c with Lem iii).

Pf: i) By L'Hopital Thm.

ii) Process by polarization: Let $\eta = \lambda$:

$$LHS \overset{\text{Sym}}{=} 2\int_0^\infty \lambda(dt)\cdot\Big[\int_{[1,t)} s\,\lambda(ds)\Big]$$

$$= -\int_1^\infty t\, d|\lambda(t,\infty)|^2 = \int_1^\infty |\lambda(t,\infty)|^2 dt.$$

from integration by part.

iii) $$\Lambda_W^*(\varphi) \overset{\text{IBP}}{=} \sup\Big\{\int_0^\infty \varphi'\cdot\varphi_\lambda'\,dt - \frac{1}{2}\|\varphi_\lambda'\|_2^2\Big\}$$

$$\leq \sup\Big\{\|\varphi'\|_2\,\|\varphi_\lambda'\|_2 - \frac{1}{2}\|\varphi_\lambda'\|_2^2\Big\}.$$

$$\overset{\|\varphi'\|_2 = \|\varphi_\lambda'\|_2}{\leq} \frac{1}{2}\|\varphi_\lambda'\|_2^2.$$

It's attained if let $\lambda(t,\infty) = \varphi'(t)$.

provided $\varphi \in H^1$.

Conversely if $\Lambda_W^*(\varphi) < \infty$. We see

$$A\phi := -\int \varphi\phi' \leq \Lambda_W^*(\varphi) + \frac{1}{2}\|\phi\|_2^2.$$

for $\forall \phi \in C_c^\infty$. $\overset{\text{Riesz}}{\Longrightarrow} \exists\, \varphi' \in L^2$.

s.t. φ' is weak derivative of φ.

Lem. (Cameron - Martin's)

For $\lambda \in H^{*}$. W^{λ} is dist. of $\theta \mapsto \theta + X_{\lambda}$.

under W. Then $W^{\lambda} \ll W$. and

$dW^{\lambda}/dW\,(\theta) := R_{\lambda}(\theta) = e^{\langle \lambda, \theta \rangle - \Lambda_W(\lambda)}$.

Pf: Set $\mathbb{P}(d\theta) = R_{\lambda}(\theta)^{-1} W^{\lambda}(d\theta)$

Next, we check $\mathbb{P} = W$. $(\Longrightarrow)$

$\int_{\Theta} e^{\langle \eta, \theta \rangle} \mathbb{P}(d\theta) = e^{\Lambda_W(\eta)}$.

Lem. (Lower bound for LDP)

$\forall G \subset H$ open. $\varliminf_{\varepsilon \to 0} \varepsilon \log(W_{\varepsilon}(G)) \geq -\inf_{G} \Lambda_W^{*}$.

Pf: Next, we use change of measure

to prove: $LHS \geq \Lambda_W^{*}(\psi)$. $\forall \psi \in G \cap H^{1}$.

Since we can find $\{\varphi_n\} \subset C_c^{\infty}$

st. $\|\varphi_n - \psi\|_{H^1} \to 0$.

So: $\Lambda_W^{*}(\varphi_n) \to \Lambda_W^{*}(\psi)$.

WLOG. assume $\psi \in C_c^{\infty}$. Set $\lambda(t, \infty)$

$= \psi'(t)$. and chose $r > 0$. st. $B(\psi, r)$

$\subset G$. Then for $0 < \delta < r$.

$$W_\varepsilon(A) \geq W_\varepsilon(B(\psi,\delta)) \overset{\text{shift}}{=} W^{-\lambda/\varepsilon^{\frac{1}{2}}}(B(0,\delta/\varepsilon^{\frac{1}{2}}))$$

$$\overset{\text{cm}}{\geq} e^{-\frac{1}{\varepsilon}(\Lambda_w(\lambda)+\delta\|\lambda\|)}\, W(B(0,\delta/\varepsilon^{\frac{1}{2}}))$$

Combine with $\Lambda_w(\lambda) = \Lambda_w^*(\psi)$.

Let $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$.

Lem: (Weak LDP).

$\psi \in \Theta$. Then: $\forall \delta > 0$. $\exists r > 0$. s.t.

$$W_\varepsilon(\bar{B}(\psi,r)) \leq \begin{cases} \exp(-1/\varepsilon\delta). & \text{if } \Lambda_w^*(\psi) = \infty \\ \exp\left(-\dfrac{\Lambda_w^*(\psi)-\delta}{\varepsilon}\right). & \text{if } \Lambda_w^*(\psi) < \infty \end{cases}$$

Pf: $W_\varepsilon(\bar{B}(\psi,r)) = W(\bar{B}(\psi/\varepsilon^{\frac{1}{2}}, r/\varepsilon^{\frac{1}{2}}))$

$$\overset{\text{chebyshev}}{\leq} \int_{\bar{B}(\psi/\varepsilon^{\frac{1}{2}},\, r/\varepsilon^{\frac{1}{2}})} e^{\langle \lambda/\varepsilon^{\frac{1}{2}},\theta\rangle}\, W(d\theta) \Big/ \inf_{\bar{B}(\psi/\varepsilon^{\frac{1}{2}},\, r/\varepsilon^{\frac{1}{2}})} e^{\langle \lambda/\varepsilon^{\frac{1}{2}},\phi\rangle}$$

$$\underset{\text{ch.f.}}{\overset{\text{c.v.}}{\leq}} e^{-\frac{1}{\varepsilon}(\langle\lambda,\psi\rangle - \Lambda_w(\lambda) - \|\lambda\| r)}, \quad \forall \lambda \in \Theta^*$$

1) If $\Lambda_w^*(\psi) = \infty$. choose $\lambda \in \Theta^*$. s.t.

$\langle \lambda, \psi\rangle - \Lambda_w(\lambda) \geq 1 + 1/\delta$. $r = 1/(1+\|\lambda\|)$

2) If $\Lambda_w^*(\psi) < \infty$. choose $\lambda \in \Theta^*$. s.t.

$\langle \lambda, \psi\rangle - \Lambda_w(\lambda) \geq \Lambda_w^*(\psi) - \delta/2$. $r = \dfrac{\delta}{2(1+\|\lambda\|)}$

And let $\ell = \inf_K \Lambda_w^*$. For $\delta > 0$:

So: $W_\varepsilon(K) \leq n\exp(-\frac{1}{\varepsilon}(\delta^{-1}\wedge(l-\delta)))$

where $K = \bigcup_1^n B(\psi_k, r_k)$. $\psi_k \in K$.

Then let $\delta \downarrow 0$.

Rmk: For general $\{\mu_\varepsilon\}_{\varepsilon>0}$ on X where

its log m.g.f. satisfies:

$\Lambda(\lambda) = \lim_{\varepsilon\to 0} \varepsilon \Lambda_{\mu_\varepsilon}(\lambda/\varepsilon)$. $\forall \lambda \in X^*$.

(Note for Wiener case: $\varepsilon\Lambda_{W_\varepsilon}(\lambda/\varepsilon)$

$= \Lambda_W(\lambda)$ & i.i.d case $n\Lambda_{\mu_n}(\frac{\lambda}{n}) = \Lambda$)

Repeat the argument of Lem. above

$\Rightarrow \overline{\lim_{\varepsilon\to 0}} \varepsilon(\log(\mu_\varepsilon(K))) \leq -\inf_K \Lambda^*(x)$.

Next, we're going to process exp. tightness:

Thm. (Fernique)

X is separable. real Frechét space. (

locally convex metrizable complete t.v.s.)

$\phi: X \to \mathbb{R}^{\geq 0}$ is subadditive. measurable &

$\phi(\alpha x) = |\alpha|\phi(x)$, $\forall \alpha \in \mathbb{R}^1$. $x \in X$. If

μ on $(X, \mathcal{B}_X)$ s.t. μ^2 satisfies it is invariant under rotation $R_{\frac{\pi}{4}} :=$

$\begin{pmatrix} \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \\ -\cos\frac{\pi}{4} & \sin\frac{\pi}{4} \end{pmatrix}$ $(R_{\frac{\pi}{4}} * \mu^2 = \mu^2)$ and $\mu(\{\phi < \infty\})$

$= 1$. Then: $\exists \alpha > 0$. s.t. $\int_X e^{\alpha \phi(x)^2} d\mu < \infty$.

Pf: For $0 < s < t$, we have:

$$\mu\{\phi \le s\} \mu\{\phi \ge t\} = \mu^2\{\phi(x_1) \le s, \phi(x_2) \ge t\}$$

$$\overset{\text{invar.}}{=} \mu^2(\{\phi(x_1 - x_2) \le 2^{\frac{1}{2}} s, \ \phi(x_1 + x_2) \ge 2^{\frac{1}{2}} t\})$$

$$\le \mu^2\{\phi(x_1) \wedge \phi(x_2) \ge 2^{-1/2}(t-s)\}$$

$$\le (\mu\{\phi(x) \ge 2^{-\frac{1}{2}}(t-s)\})^2$$

Choose s. st. $\mu\{\phi < s\} > \frac{1}{2}$ and $\{t_n\}$

defined by $t_0 = s$. $t_n = s + 2^{\frac{1}{2}} t_{n-1}$.

$$\Rightarrow \mu\{\phi \ge t_n\} / \mu(\phi \le s) \le (\mu(\phi \ge s) / \mu\{\phi \le s\})^{2^n}.$$

$\Rightarrow$: $\mu\{\phi^2 \ge 2^n \beta\} \le e^{-2^n \sigma}$. $\exists \beta, \sigma > 0$.

Then, we can choose $\alpha < \sigma / 2\beta$.

$$\int e^{\alpha \phi^2} d\mu = \int_{\{\phi^2 \ge \beta\}} + \int_{\{\phi^2 \le \beta\}}$$

$$\le C + \sum_{n \ge 1} e^{2^{n+1}\alpha\beta} \mu\{2^n \beta \le \phi^2 \le 2^{n+1}\beta\} < \infty.$$

Lm. Set $\phi(\theta) = \sum_{n\geq 1} 2^{-n} \sup_{1\leq s<t\leq n} \frac{|\theta(t)-\theta(s)|}{|t-s|^{\frac{1}{8}}} + \sup_{t\geq 1} \frac{|\theta(t)|}{t^{3/4}}$

Then: $\{\phi(\theta) \leq L\} \underset{cpt}{\subset} \Theta \quad \forall L>0$. And $\exists$

$\alpha>0$. st. $\int_\Theta e^{\alpha\phi(\theta)^2} dW(\theta) < \infty$.

Cor. $K_L = \{\phi^2(\theta) \leq L/\alpha\} \underset{cpt}{\subset} \Theta$ and $\{K_L\}$

Satisfies: $\overline{\lim}\, \varepsilon \log(W_\varepsilon(K_L^c)) \leq -L$.

Pf: By Chebyshev inequi.: $W_\varepsilon(K_L^c) =$

$W(\{\alpha\phi^2(\theta) \geq L/\varepsilon\}) \leq e^{-L/\varepsilon} \int e^{\alpha\phi^2(\theta)} dW$

$\leq C e^{-L/\varepsilon}$

Pf: i) $\{\theta(t)\}_{\phi(\theta)\leq L}$ Satisfies $\|\theta\|_{C^{\frac{1}{8}}} \leq L$

$\sup_{t\geq 1} \square \Rightarrow \{\theta\}_\square$ is uniform bdd $\Rightarrow$ Ascoli

ii) W is Invar. under $\forall R_\theta$. To use the Fernique Thm. We prove: $W\{\phi<\infty\} = 1$

$\Leftarrow \mathbb{E}_W(\phi(\theta)^8) < \infty$. $(\int \|\theta\|^8_{C^{1/4}[0,n]} dW$

$\overset{\text{Scal.}}{=} n^2 \int \|\theta\|^8_{C^{\frac{1}{8}}[0,1]} dW \sim n^2$. $\sup_{t\geq 1} \frac{|\theta(t)|}{t^{3/4}}$

$\leq \sum_n \left(\sup_{n\leq t\leq n+1} \frac{|\theta(t)-\theta(n)|}{n^{3/4}} + |\theta(n)|/n^{\frac{3}{4}} \right))$

Thm. (Schilder's)

$-\inf_{\Gamma^o} \Lambda_W^* \leq \underline{\lim}_{\varepsilon\to 0} \varepsilon \log W_\varepsilon(\Gamma) \leq \overline{\lim}_{\varepsilon\to 0} \varepsilon \log W_\varepsilon(\Gamma) \leq -\inf_{\overline{\Gamma}} \Lambda_W^*$

for $\forall \Gamma \in \mathcal{B}_\Theta$.