General Cramer Theory

Set $(E, \mathcal{B})$ is a measure space. $I : E \to \mathbb{R}^{\geq 0}$

s.t. $\forall L > 0$. $\{I(x) \leq L\} \underset{cpt}{\subset} E$. good rate func.

<u>Def</u>: i) LDP with rate func. I for $\{\mu_\varepsilon\}_{\varepsilon>0}$

is: For $\forall G \subset E$ open. $\forall F \subset E$ closed.

$$\underline{\lim}_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \geq -\inf_G I$$

$$\overline{\lim}_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(F) \leq -\inf_F I$$

ii) Weak LDP with rate I if $\forall K \underset{cpt}{\subset} E$

$: \overline{\lim}_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(K) \leq -\inf_K I$.

Recall tightness of $\{\mu_\varepsilon\}_{\varepsilon>0} \Leftrightarrow \{\mu_\varepsilon\}$ is cpt in $(\mathcal{P}_+, \text{weak-top.})$. And we can choose $K_\delta \underset{cpt}{\subset} E$. s.t. $\sup_\varepsilon \mu_\varepsilon(K_\delta^c) < \delta$. Next. we extend it to exponential type:

<u>Def</u>: $\{\mu_\varepsilon\}$ satisfies exponential tightness

if $\forall \delta > 0$. $\exists K_\delta \subset\subset E$. s.t. We have:

$$\overline{\lim}_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(K_\delta^c) \leq -1/\delta.$$

Claim: weak LDP + exp. tightness $\Rightarrow$ strong upper bound.

Rmk: We want to restrict on cpt set K_δ.

Pf: $\varepsilon \log \mu_\varepsilon(F) \le \varepsilon \log(\mu_\varepsilon(F \cap K_\delta) + \mu_\varepsilon(K_\delta^c))$

$\le -\inf_{F \cap K_\delta} I \vee -1/\delta + \varepsilon \log 2$. Let $\delta \downarrow 0$.

Challenge is that now in Banach space E. s.t. $\dim E = +\infty$. Then $\overline{B}(x, R)$ isn't cpt.

Rmk: But sometimes $\exists \|\cdot\|_1 >> \|\cdot\|$. s.t. $\overline{B}$ may be cpt in $(E, \|\cdot\|_1)$. eg.:

$E = C([0,1]) \cap \{g(0) = 0\}$ with $\|\cdot\|_\infty$.

$\{\|g\|_\infty = 1\}$ isn't cpt. But if we consider $\|\cdot\|_\alpha$, α-Hölder norm. then:

$\{\|g\|_\alpha \le 1\} \underset{\text{cpt}}{\subset} E$.

If $\exists \beta > 0$. s.t. $\Lambda(\beta) = \overline{\lim_{\varepsilon \downarrow 0}} \varepsilon \log \mathbb{E}_{\mu_\varepsilon}(\exp(\beta \|X\|_\alpha / \varepsilon)) < \infty$. Then we can

Conclude it satisfies exp. tight :

$K_\delta := \{ \|x\|_\alpha \le 1/\delta \} \subseteq_{cpt} E$.

$\mu_\varepsilon(K_\delta^c) \overset{\text{Chebyshev}}{\le} \mathbb{E}_{\mu_\varepsilon}(\exp(\beta \|x\|_\alpha/\varepsilon)) / e^{\beta/\varepsilon\delta}$.

$\Rightarrow \varepsilon \log \mu_\varepsilon(K_\delta^c) \le \varepsilon \log \Lambda^\varepsilon(\beta) - \beta/\delta$.

Rmk: It's kind of Fernique Thm.

As for weak LDP, we claim it can come from : $\forall x \in E$, $I(x) < \infty$, then,

$\lim_{\delta \downarrow 0} \overline{\lim_{\varepsilon \to 0}} \varepsilon \log \mu_\varepsilon(\overline{B}(x, \delta)) \le -I(x)$.

Pf: Note $K \subset\subset E$. $\forall \delta > 0$, $\exists \bigcup_1^n B(x_i, \delta) \supseteq K$.

$\Rightarrow \varepsilon \log(\mu_\varepsilon(K)) \le \varepsilon \log(\sum_1^n \mu_\varepsilon(B(x_i, \delta)))$

$\le \varepsilon \log n + \max_i \varepsilon \log \mu_\varepsilon(B^i(\delta))$

We've coped with upper bound above. And for lower bound, It's from Varadhan Lem :

Lem. $\{\mu_\varepsilon\}_{\varepsilon > 0}$ satisfies LDP with rate $I(x)$

If $F : E \to \overline{\mathbb{R}}$ is conti. Then :

$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_{\mu_\varepsilon}(e^{F(x)/\varepsilon}) = \sup_E (F(q) - I(q))$

Cor. If $I(x)$ is convex. Then:

$I(x) = \Lambda^*(x)$. $*$ is Legendre transf.

Pf: Set $F(x) = \langle \lambda, x \rangle$.

We have RHS above $= \Lambda^*(x)$.

Rmk: For $\Lambda(\lambda)$ convex. $\Rightarrow \Lambda^*$ convex.

if additionally $\Lambda(\lambda)$ is l.s.c.

$\Rightarrow \Lambda^{**} = \Lambda$.

Contraction Principle:

For (μ_ε) LDP with I. $\phi \in C(E, \tilde{E})$.

Set $\tilde{\mu}_\varepsilon = \mu_\varepsilon \circ \phi^{-1}$. Then: $\tilde{\mu}_\varepsilon$ satisfies

LDP with $\tilde{I}(y) = \inf\limits_{x : \phi(x) = y} I(x)$.

Rmk: If ϕ isn't conti. Then we need

some exponential approxi. $(\phi_\delta(x))_{\delta > 0}$

$\subset C(E, \tilde{E})$. s.t. $\forall \xi > 0$

$\overline{\lim\limits_{\varepsilon \to 0}} \varepsilon \log \mu_\varepsilon (\| \phi_\delta - \phi \| > \xi) = -\infty$.

Variance: (For lower bound)

Consider $L_N(x) = \frac{1}{N}\sum_1^N \delta_{X_k}$. $\mu_N = (L_N)_* \mathbb{P}$.

Next, we assume (X_k) is stationary. Then:

by ergodic Thm, $L_N \xrightarrow{w} \mu$. ($X_k \sim X \sim \mu$.)

i.e. $\lim_{N\to\infty} \mathbb{P}_N(L_N \in B(\mu, \varepsilon)) = 1$. $\forall \varepsilon > 0$.

($\mathbb{P}_N$ is p.m. restricted on $\sigma(\{X_k\}_1^N)$)

The idea to derive its LDP is to find

Q_N. st. $Q_N \sim \mathbb{P}_N$ and Q_N is stationary for

$\theta \in L$. i.e. $\exists \varepsilon > 0$. st. $Q_N(L_N \in B(\theta, \varepsilon)) \to 1$.

We set $F_N = dQ_N / d\mathbb{P}_N$. $H(Q \mid \mathbb{P}) := \lim_{N\to\infty} \frac{H(Q_N \mid \mathbb{P}_N)}{N}$

Where $H(\nu \mid \mu) = \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} \, d\mu$

Rmk: For X_k i.i.d. we have $H(Q_N \mid \mathbb{P}_N) =$

$N H(Q \mid \mathbb{P})$.

1) $\log(\mathbb{P}_N(L_N \in B(\theta, \varepsilon)) / Q_N(L_N \in B(\theta, \varepsilon)))$

$= \log[(\int_B F_N^{-1} \, dQ_N(\cdot \mid L_N \in B(\theta, \varepsilon))]$.

$\overset{\text{Jensen}}{\underset{\log x = -\log t}{\geq}} \int_B (\log F_N^{-1}) \, dQ_N(\cdot \mid B) \overset{\text{take}}{\underset{E(\cdot)}{=}} -\int_B \log F_N \cdot F_N \, d\mathbb{P}_N$.

$$= \int -\varphi(L_N)\, d\mathbb{P}_N - \int \varphi(L_N)\, I_{\{L_N \in B\}}\, d\mathbb{P}_N$$

$$\geq -H(Q_N \| \mathbb{P}_N) - 1/\varepsilon. \quad (-\varphi(t) \geq -1/\varepsilon.)$$

ii) So $\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N(L_N \in G) \geq$

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N(L_N \in B(\theta, \varepsilon)) \geq$$

$$\lim_{N\to\infty} \frac{1}{N} \{ -H(Q_N \| \mathbb{P}_N) - 1/\varepsilon - \log \theta_N(\cdots) \}$$

$$= -H(Q \| \mathbb{P})$$

Rmk: It can be applied in SDE by Girsanov

Thm (which derive the shifted p.m.)