Finance in Conti Time.

(1) <u>Geometric BM</u>:

Consider $X_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$. By Itô's:
(B_t is SBM)

$$dX_t = X_t(\mu\,dt + \sigma\,dB_t)$$

We call such X_t by Geometric BM.

<u>Rmk</u>: $\log X_t$ is normally distributed.

Bachelier first used BM to model the price development in economics. But it's inadequate:

i) it can be negative

ii) One should think in term "return" rather than "price".

But we can use the geometric one:

$dX_t / X_t = \mu\,dt + \sigma\,dB_t$.

i) $dX_t / X_t = (X_{t+dt} - X_t)/X_t$. the gain per unit of value in stock. i.e. return.

ii) $\mu\,dt$ deterministic term. is the riskless investment in bank with interest rate μ.

iii) $\sigma\,dB_t$ the random term models uncertainty in market.

Return intervals:

i) Long (Macroscopics): invest in time-scale of month / year. So the price-change is the sum of price-change over days and these are indept. By CLT $\Rightarrow$ normal dist.

ii) Intermediate (Mesoscopics): in scale of day. It has fatter tail so normality won't come in. The model we use is hyperbolic dist.

iii) Short (Microscopics): High-frequency data is available nowadays from Internet. So the tail is much fatter if we consider the interval of order of seconds.

(2) Black-Scholes Model:

Consider: $dB_t = r_t B_t dt$ models the riskless invest in bank with interest rate r_t. $\Rightarrow B_t = B_0 e^{\int_0^t r_s ds}$.

and : $dS_t = \mu_t S_t dt + \sigma S_t dW_t$. models the risky invest in stock. it's GBM-like. (W_t is SBM)

Lemma: $S_t = S_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t + \int_0^t \mu_s ds}$.

Pf. By Itô's formula.

Set $\tilde{S}_t = S_t / B_t$ the discounted stock price.

$$\Rightarrow d\tilde{S}_t = \tilde{S}_t (\mu_t - r_t) dt + \sigma \tilde{S}_t dW_t.$$

$$\overset{\Delta}{=} \sigma \tilde{S}_t (\theta_t dt + dW_t) . \quad \theta_t =: \frac{\mu_t - r_t}{\sigma}.$$

Apply Girsanov Thm. we can find the risk neutral p.m. $d\mathbb{P}^* = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds} d\mathbb{P}$.

which shift : $\theta_t \mapsto 0$.

<u>Thm.</u> (Risk-neutral Valuation Formula)

The no-arbitrage price of claim $h(S_T)$ is

$$F(t, x) = \mathbb{E}^*_{t,x} (\frac{B_t}{B_T} h(S_T) \mid \mathcal{F}_t). \quad S_t = x.$$

Besides, under $\mathbb{P}^*$. $dS_t = r_t S_t dt + \sigma S_t dW_t$.

<u>Pf:</u> $d\tilde{V}_t = H_t d\tilde{S}_t = H_t \cdot \sigma \tilde{S}_t dW_t$ under $\mathbb{P}^*$.

where H_t is perfect hedging of h.

$\Rightarrow \tilde{V}_t$ is $\mathbb{P}^*$-mart. as before.

<u>Thm.</u> (Conti. BS Formula)

The value of European call option with the striking price K and expire time T at $t=t$ is $C(t, S_t, K, T) = S_t \phi(z) - \frac{K B_t}{B_T} \phi(z - \sigma\sqrt{T-t})$

where $z = (\log(S_0/K) + \frac{1}{2}\sigma^2(T-t) + \int_t^T r_s) / \sigma\sqrt{T-t}$

Pf: By Thm above. WLOG. set $t = 0$. $S_0 =$

$$C(0, S_0, K, T) = \mathbb{E}^*(\tilde{S}_T - K/B_T)_+ .$$

$$\tilde{S}_T \sim S_0 e^{\int_0^T r_s ds - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T} Z}, \quad Z \sim N(0,1).$$

① BS PDE:

Apply Itô's formula on $B_t^{-1} C(t, S_t)$. (Note $C(t,x) \in C^{2,1}$).

$$\Rightarrow B_t \, d(B_t^{-1} C(t, S_t))$$

$$= \partial_x C(t,S_t)\sigma S_t \, dW_t + \Big(\partial_x C(t,S_t) r_t dS_t + \frac{\sigma^2 S_t^2}{2}\partial_{xx} C(t,S_t)$$

$$+ \partial_t C(t,S_t) - r_t C(t,S_t)\Big) dt$$

Since $\tilde{F}(t, S_t) = C(t,S_t)/B_t$ is $\mathbb{P}^*$-mart.

So: $-r_t C(t,x) + \partial_t C(t,x) + r_t x \, \partial_x C(t,x) + \frac{\sigma^2 x^2}{2}\partial_{xx} C(t,x) = 0$

with boundary cond.: $C(T,x) = (X - K)_+$.

Rmk: For general continengent claim $F(S_T)$.

We can change the boundary condition:

$C(T, x) = F(x)$.

② Hedging:

i) By BS-PDE: $dC(t,S_t) = \Big(-\frac{S_t \partial_x C(t,S_t)}{B_t} + \frac{C(t,S_t)}{B_t}\Big) dB_t$

$$+ \partial_x C(t,S_t) \, dS_t .$$

Thm $\forall t \le T$. $H_t^0 = \partial_x C(t, S_t)$.

$H_t = (-S_t \partial_x C(t, S_t) + C(t, S_t)) / B_t$

is the perfect hedging for the call option

Pf: easy to see H is SF.

ii) For $h \in \mathcal{F}_T$. $h \in L^2(\mathbb{P})$. If H is hedging:

$\tilde{V}_t = H_t^0 + H_t \tilde{S}_t = E^*((B_T)^{-1} h \mid \mathcal{F}_t) \hat{=} M_t$.

By mart. representation: $M_t = m_0 + \int_0^t k_s dW_s$.

with Ocone - Clark formula, we have:

$k_s = (B_T)^{-1} \mathbb{E}(D_s h \mid \mathcal{F}_s)$

$= (B_T)^{-1} \mathbb{E}(h'(S_T) \sigma S_T \mid \mathcal{F}_s)$ if $h = h(S_T)$.

Since $S_T = e^{c + \sigma W(I_{[0,T]})}$ c is const.

Set $\begin{cases} H_t = B_t k_t / \sigma S_t \\ H_t^0 = M_t - H_t \tilde{S}_t \end{cases}$ the hedging.

$\Rightarrow dM_t = \frac{k_t}{\sigma \tilde{S}_t} \cdot \sigma \tilde{S}_t dW_t = H_t d\tilde{S}_t$ under $\mathbb{P}^*$.

(3) Barrier Option:

Def. The Barrier option with expire time T and payoff function $\eta = I\{\max_{[0,T]} S_t \ge K\}$.

Consider model : $\begin{cases} B_t = B_0 e^{rt} & \text{deterministic} \\ S_t = S_0 e^{(r-\sigma^2/2)t + \sigma W_t} \end{cases}$.

$\Rightarrow$ The arbitrage value of Barrier option is :

$$V_0 = e^{-rT} \mathbb{P}_0 (\max_{[0,T]} S_t \geq k)$$

$$= e^{-rT} \mathbb{P}_0 (\max_{[0,T]} (W_t + (\tilde{r} - \frac{\sigma}{2}) t) \geq \tau) .$$

where $\tilde{r} = r/\sigma$. $\tau = (\log k)/\sigma$.

Recall : Set $\theta = -(\tilde{r} - \frac{\sigma}{2})$. under $\mathbb{P}_\theta \overset{\Delta}{=}$ $e^{\theta W_t - \frac{\theta^2}{2} t} \mathbb{P}_0$. $\widetilde{W_t} = W_t + (\tilde{r} - \frac{\sigma}{2}) t$ has same law as W_t under $\mathbb{P}_0$.

So : $\mathbb{P}_0 (\max_{[0,T]} S_t \geq k) = e^{\frac{1}{2}(\theta^2 - 2\theta r + \theta) T} \mathbb{E}_{\mathbb{P}_\theta} (e^{-\theta \widetilde{W}(T)} I_{\{\max_{[0,T]} \widetilde{W}(t) \geq \tau\}})$

$$= e^{\square} \mathbb{E}_{\mathbb{P}_0} (e^{-\theta W(T)} I_{\{\max_{[0,T]} W(t) \geq \tau\}})$$

$\Rightarrow$ Apply Reflection principle, we can obtain V_0.

<u>Rmk</u>: For Kockin's option (put) with payoff at T : $(K - S_T)_+ I_{\{\max_{[0,T]} S_t \geq H\}}$. $H > k$.
We can also find $V_0(k)$ as above.

(4) <u>American put option in $[0,\infty]$:</u>

Recall we use snell envelop to solve it on finite time-horizon. Next, we consider it on $[0, \infty]$.

Consider under $\mathbb{P}^*$: $dX_t = rX_t dt + \sigma X_t dW_t$.

For stopping time τ. Solve: $V(x) = \sup_{\tau} \mathbb{E}_x(e^{-r\tau}(K-X_\tau)_+)$.

which is the value of option.

Denote $L_x = rx\,\partial_x + \frac{\sigma^2 x^2}{2}\partial_{xx}$. generator of X.

Set $\tau_b = \inf\{t \geq 0 \mid X_t \leq b\}$. $b < K$.

Rmk: We will stop at τ_b. Since the closer X to 0. We're less likely to profit

It converts to solve:

$L_x V = rV$. for $x > b$; $V(b) = (K-b)_+$ (b.c.).

$V'(x) = -1$ for $x = b$. (Smooth fit)

Besides, $V(x) > (K-x)^+$. $\forall x > b$;

$V(x) = (K-x)_+$. $\forall x < b$.

Set $\lambda = \sigma^2/2$. $\Rightarrow$ Solve: $\lambda x^2 V'' + rxV' - rV = 0$.

$\Rightarrow$ We have $V(x) = C_1 x + C_2 x^{-r/\lambda}$. C_i: const.

But V is bdd $\Rightarrow$ $V(x) = C x^{-r/\lambda}$.

with $V'(b) = -1$. We have $C = \frac{\lambda}{r}\left(\frac{K}{1+\lambda/r}\right)^{1+r/\lambda}$.

So: $$V(x) = \begin{cases} K - x & . \quad x \leq b \\ \frac{\lambda}{r}\left(\frac{K}{1+\lambda/r}\right)^{1+r/\lambda} x^{-r/\lambda} & . \quad x > b. \end{cases}$$

Rmk: It's applicable in real option, which concerns with the business-decision making.