Insurance Math

- The fire insurance starting in 1681 is the first insurance due to the great fire in London.

- Next, we will use compound Poisson process to model the arrivals of claim with random claim sizes.

  Suppose the rate of claim arrival is λ.

  Each claim (X_k) are i.i.d. with mean μ.

  So the total value of claim in $[0, t]$ is:

  $S(t) := \sum_{1}^{N(t)} X_k$. where $N(t) \sim Poi(\lambda t)$. $X_k \sim F$

  Rmk: i) The main weakness here is "indept".

  ii) By Wald's: $\mathbb{E}(S(t)) = \lambda \mu t$.

- Next, we consider the Ruin Problem:

  The initial capital from the company is u.

  The premium income comes in from the policy holders at const. rate c.

  $\Rightarrow$ The total capital is $ct + u$ at time $= t$.

Denote W_i is the inter-claim waiting time.

$Z_i \stackrel{\Delta}{=} X_i - cW_i$. $\mathbb{E}(Z_i) = \mu - c/\lambda$.

By SLLN, to avoid bankruptcy, we require:

$\mathbb{E}(Z_i) < 0$. i.e. $c > \lambda\mu$ ... (NPC)

We call it Net profit condition (NPC).

<u>Premium Calculation:</u>

First, we evaluate safety loading (SL) $e > 0$.

Set $c = (1+e)\mathbb{E}(X_k) / \mathbb{E}(W_k) = (1+e)\lambda\mu$.

<u>Rmk:</u> $e = \dfrac{c - \lambda\mu}{\lambda\mu} > 0 \iff$ NPC holds.

① <u>Lundberg's Inequi.:</u>

Set $M(s) := \mathbb{E}(e^{sX_i})$. Assume:

i) Small claim condition (SCC). $\exists s_0 > 0$. s.t.

$M(s) < \infty$. for $\forall |s| < s_0$.

<u>Rmk:</u> i) It implies $\mathbb{P}(X_i > x) \leq M(s)/e^{sx}$. decay exponentially on tail prob.

ii) $M(s)$ is smooth, convex. $M(0) = 1$.

ii) Lundberg coefficient $r > 0$. exists. defined by

$r > 0$. $M_{Z_i}(r) = \mathbb{E}(e^{r(X_i - cW_i)}) = 1$. i.e.

$M(r) = 1 + \dfrac{cr}{\lambda}$.

Rmk: The bigger r means the bigger so.

$\Rightarrow$ The better.

Thm. If NPC, SCC, LC holds. Then the ruin prob. $\psi(u)$ with initial capital u satisfy:

$$\psi(u) \leq e^{-ur}.$$

Pf: Denote $S_n = \sum_1^n Z_k$.

$$\psi_n(u) := \mathbb{P}(\max_{1\leq k\leq n} S_k > u) \uparrow \psi(u).$$

$\Rightarrow$ Next, we prove: $\psi_n(u) \leq e^{-ru}$ $\forall n$.

by induction on n.

$n=1$. by Chebyshev: $\psi_1(u) \leq M_{Z_1}(r)/e^{ru} = e^{-ru}$

$k=n+1$: By one-step argument:

$$\psi_{n+1}(u) = \mathbb{P}(Z_1 > u) + \mathbb{P}(Z_1 \leq u, \max_{2\leq k\leq n+1} -(S_k - Z_1) > u - Z_1) \stackrel{\Delta}{=} P_1 + P_2.$$

$$P_1 \leq \int_{(u,\infty)} e^{r(x-u)} dF(x)$$

$$P_2 \stackrel{MP.}{=} \int_{(-\infty,u]} \mathbb{P}(\max_{1\leq k\leq n} (X + S_k) > u) dF(x)$$

$$= \int_{(-\infty,u]} \psi_n(u-x) dF(x)$$

$$\stackrel{hypo.}{\leq} \int_{(-\infty,u]} e^{r(u-x)} dF(x).$$

So: $\psi_{n+1}(u) \leq e^{-ru} M(r) = e^{-ru}$.

Set $n\to\infty$. So: $\psi(u) \leq e^{-ru}$.

② Renewal Equation:

Recall renewal equation for $\tilde{F}$ and $z_0(t)$ is

$$Z(t) = z_0(t) + \int_0^t Z(t-u)\, d\tilde{F}(u).$$

Next, we will show $\psi(u)$ almost satisfies renewal-type equation.

Note: $\mu = \mathbb{E}(X_1) = \int_0^\infty (1-F(x))\,dx$.

Set $dG(x) = \frac{1-F(x)}{\mu}\,dx$ is also a density.

$\phi(u) = 1-\psi(u)$. the prob. of nonruin.

$$\phi(u) = \mathbb{E}\left(I_{\{Z_1 \le u\}}\, \mathbb{E}\left(I_{\{S_n - Z_1 \le u - Z_1,\ \forall n \ge 2\}} \mid Z_1 \right)\right)$$

$$\overset{\text{MP.}}{=} \int_0^\infty \lambda e^{-\lambda w}\, dw \int_0^{u+cw} \phi(u+cw-x)\, dF(x)$$

$$\overset{z=u+cw}{=} \frac{\lambda e^{\lambda u/c}}{c} \int_u^\infty e^{-\lambda z/c}\, g(z)\, dz.$$

where $g(z) = \int_0^z \phi(z-x)\, dF(x)$

Diff

$$\Rightarrow \phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-x)\, dF(x).$$

integrate

$$\Rightarrow \phi(t) - \phi(0) = \frac{\lambda}{c}\int_0^t \phi\, du - \frac{\lambda}{c}\phi(0)\int_0^t F + \frac{\lambda}{c}\int_0^t du \int_0^u F(x)\phi(x-u)\,dx.$$

$$= \frac{\lambda}{c}\int_0^t \phi(t-x)(1-F(x))\,dx$$

$$= (1+\epsilon)^{-1}\int_0^t \phi(t-x)\, dG(x).$$

Note: $\phi(u) \uparrow 1$ as $u \to \infty$. by SLLN. $S_n \to -\infty$.

Set $u \to \infty$. We can find $\phi(0) = \frac{c}{1+c}$.

So: $\phi(u) = \frac{c}{1+c} + \frac{1}{1+c} \int_0^u \phi(u-x) \frac{1-F(x)}{\mu} dx$

i.e. $\psi(u) = \frac{1}{1+c} \int_u^\infty \frac{(1-F(x))}{\mu} dx + \frac{1}{1+c} \int_0^u \psi(u-x) \frac{1-F}{\mu} dx$

③ Cramer's estimate of ruin:

By LC and integrate by part: $\int_0^\infty (1-F) e^{rx} = \frac{c}{\lambda} = \mu(1+c)$

$\Rightarrow \frac{\lambda}{c}(1-F(x)) e^{rx} dx$ is density on $\mathbb{R}^{\geq 0}$. Which is called Esscher transform.

Thm. Under NPC. SCC. LC. We have:

$$e^{ru} \psi(u) \xrightarrow{u \to \infty} \text{Const.} = (c-\lambda\mu) / \lambda r \int_0^\infty x e^{rx} (1-F(x)) dx.$$

Pf: It's direct from key Renewal Thm.

$\lim_{t \to \infty} Z(t) = \tilde{\mu}^{-1} \int_0^\infty z_0(t)$. $\tilde{\mu} = E(\tilde{F})$.

So we only need to calculate:

$$\int_0^\infty e^{ru} du \int_u^\infty (1-F(x)) dx = \frac{c-\lambda\mu}{\lambda r}$$

follows from integrate by part.

Besides, $\tilde{\mu} = \frac{\lambda}{c} \int_0^\infty x e^{rx} (1-F(x)) dx$

follow from the observation below.

Note $\psi(u) e^{ru}$ satisfies (RE):

$$(\psi(u) e^{ru}) = e^{ru} \int_u^\infty \frac{(1-F(x))}{(1+c)\mu} dx + \int_0^u (\psi(u-x) e^{r(u-x)}) \frac{e^{rx}(1-F(x))}{(1+c)\mu} dx.$$