Foreign Exchange.

- The market for derivative securities whose payoff depend on exchange rate exist, because there's uncertainty about future rate of exchange between diff. currencies.

- Consider Y_t is rate of exchange from pounds to dollar at time t. (i.e. Y_t pounds $\leftrightarrow$ 1 dollar)

  A_t, B_t are share prices of US dollar and British pounds [resp.]. S.t.
  $$\begin{cases} dA_t = r_A(t) A_t dt \\ dB_t = r_B(t) B_t dt \\ A_0 = B_0 = 1. \end{cases}$$
  r_A, r_B are interest rate.

<u>Prop.</u> If Q_B is risk-neutral p.m. for pound investors. Y_t solves: $dY_t = \mu_t Y_t dt + \sigma_t Y_t dW_t$ under Q_B, where W_t is SBM. Then:

$\mu_t = r_B(t) - r_A(t)$.

So: $Y_t = Y_0 e^{\int_0^t (r_B(s) - r_A(s) - \sigma_s^2/2) ds + \int_0^t \sigma_s dW_s}$.

<u>Pf:</u> Discounted value for British pound is: $A_t Y_t / B_t = Y_0 e^{\int_0^t (r_A - r_B + \mu) ds} \cdot e^{\int_0^t \sigma_s dW_s - \frac{1}{2}\sigma_s^2 ds} =: M_t \cdot f(t)$

$M_t = e^{\int_0^t \sigma_s dW_s - \square}$ is mart. $\Rightarrow$ $f(t) \equiv$ const.

Lemma For Q_A, Q_B are risk-neutral prob. measures for dollar and pound investors. Then:

$$Q_A \sim Q_B \text{ and } \left.\frac{dQ_A}{dQ_B}\right|_{\mathcal{F}_T} = e^{\int_0^T \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt}$$

Pf: For claim with value V_t at time t in dollars. $\Rightarrow U_t = V_t Y_t$ is its value in pound.

Since
$$\begin{cases} U_0 = B_T^{-1} \mathbb{E}_{Q_B}(U_T) \\ V_0 = A_T^{-1} \mathbb{E}_{Q_A}(V_T). \end{cases}$$

$$\Rightarrow \mathbb{E}_A(V_T) = \mathbb{E}_B(V_T \cdot \frac{Y_T}{Y_0} e^{\int_0^T (r_A - r_B) dt}),$$

prop. Under condition above. If Y_t solves SDE:

$dY_t = \mu_t Y_t dt + \sigma_t Y_t dW_t$ under Q_A. Then:

$$\mu_t = r_B(t) - r_A(t) + \sigma_t^2.$$

Pf: $Y_t = Y_0 e^{\int_0^T (r_B - r_A + \sigma_t^2/2) dt + \sigma_t d\tilde{W}_t}$.

under Q_A. $\tilde{W}_t = W_t - \int_0^t \sigma_t \sim W_t$ under Q_B.

by change of measure in Lem. above.

Rmk: Under diff. risk-neutral p.m. The drift coefficient μ won't agree. which depends on exchange rates.

It seems paradoxical. But Q_A, Q_B aren't the real p.m. for actual model. They only work when we calculate the arbitrage price.

Consider the option (call) that gives the owner right to exchange 1 dollar to K pounds at expire time T.

<u>Denote:</u> V_t is value in dollar at time $= t$ for the option.

<u>Rmk:</u> $V_T = (K/Y_T - 1)^+$.

$$\Rightarrow V_0 = e^{-\int_0^T r_A(t)dt}\, \mathbb{E}_A\left(K Y_0^{-1} e^{\int_0^T (r_A - r_B + \sigma_t^2/2)dt - \sigma_t dW_t} - 1\right)^+$$

is the arbitrage price in dollar.