Malliavin Derivatives.

- The Malliavin calculus is infinite-dim calculus on Wiener space, aiming to give a prob. proof of Hörmander's theorem originally.

Next, we fix a H-isonormal process W.

(1) Definitions and properties:

① Set the underlying prob. space is $(\Omega, \mathcal{F}, \mathbb{P})$.

Def: i) $C_p^\infty(\mathbb{R}^d) := \{ h \in C^\infty \mid \forall \frac{\partial^\alpha}{\partial x^\alpha} h \text{ has poly growth} \}$.

$C_b^\infty(\mathbb{R}^d) := \{ h \in C_p^\infty \mid \forall \frac{\partial^\alpha}{\partial x^\alpha} h \text{ is bdd} \}$.

$C_c^\infty(\mathbb{R}^d) := \{ h \in C_b^\infty \mid \text{Supp } h \text{ is cpt} \}$.

ii) $\mathcal{S} := \{ X = f(W(h_1) \cdots W(h_n)),\ f \in C_p^\infty(\mathbb{R}^n) \}$.

and def $\mathcal{S}_b$, $\mathcal{S}_c$ similarly.

Rmk: $\mathcal{S}_c \subset \mathcal{S}$, $\mathcal{P}_0 \subset \mathcal{S}$. all dense in $L^p(\Omega)$

for $\forall p \geq 1$

iii) $X = f(W(h_1) \cdots W(h_n)) \in \mathcal{S}$. The Malliavin derivative of X is:

$$DX =: \sum_1^n \partial_j f(W(h_1) \cdots W(h_n))\, h_j. \quad \text{a.s.}$$

Rmk: $\forall h \in H$, we have:

$$\langle DX, h\rangle = \lim_{t\to 0} \big(f(W(h_1)+th_1, \cdots W(h_n)+th_n) - f(W(h_1), \cdots W(h_n)) \big)/t$$

e.g. $D B_t^* = D(W(I_{[0,t^*]})) = I_{[0,t^*]}$.

where t^* is a.s.-unique point

since $\mathbb{P}(B_t - B_s = 0) = 0$.

Lemma. (Well-def)

If $X \in \mathcal{S} = f(W(h_1) \cdots W(h_n))$

$= g(W(e_1) \cdots W(e_m))$

where (e_k) is o.n.b of $\{h_k\}$

Then: $\sum_1^n \partial_j f(W(h_1) \cdots W(h_n)) h_j = \sum_1^m \partial_j g(W(e_1) \cdots) e_j$.

Pf: WLOG. Let $\mathrm{span}\{h_k\} = \mathrm{span}\{e_j\}$.

Otherwise. Let $\tilde{f}(x_1 \cdots x_{n+m}) = f(x_1 \cdots x_n)$.

and $\tilde{g}(x_1 \cdots x_{n+m}) = g(x_1 \cdots x_m)$.

Note: $h_j = \sum_1^n \langle h_j, e_k\rangle e_k$. $\Rightarrow \exists T$. linear.

s.t. $T(W(e_1) \cdots W(e_m)) \overset{a.s.}{=} (W(h_1) \cdots W(h_n))$.

So: $f \circ T \equiv g$. (Or $|f\circ T(x_0) - g(x_0)| > 0$ on $U(x_0)$, $\mathbb{P}(U(x_0)) > 0$)

RHS $= \sum \partial_j (f\circ T)(\cdots) e_j = \sum \partial_j f(\cdots) T e_j$

$= \sum \partial_j f(W(h_1) \cdots) h_j$.

Lemma. (Integration-by-part)

$X \in \mathcal{S}$ and $h \in H$. $\Rightarrow \mathbb{E}\langle DX, h\rangle = \mathbb{E}(X\,W(h))$.

Rmk: It's intuitive: $\langle DX, h\rangle = \langle X, \delta h\rangle = X\,W(h)$.

Pf: WLOG. $\|h\| = 1$. $X = f(W(e_1), \cdots W(e_n))$. s.t.

(e_i) is o.n.b and $e_1 = h$.

$\Rightarrow \langle DX, h\rangle = \partial_1 f(W(e_1) \cdots W(e_n))$.

Denote $\varphi(x) = (2\pi)^{-n/2} e^{-\frac{1}{2}\sum_1^n x_k^2}$.

So. $LHS = \int \partial_1 f(x)\,\varphi(x)$

$= -\int f(x)\,\partial_1\varphi(x) = RHS$.

Cor. $X, Y \in \mathcal{S}$. $h \in H \Rightarrow \mathbb{E}(Y\langle DX, h\rangle) =$

$\mathbb{E}(XY\,W(h) - X\langle DY, h\rangle)$.

Pf: Check: $D(XY) = XDY + YDX$.

② **Properties:**

Prop. $p \geq 1$. The Malliavin derivative: D:

$L^p(\Omega) \to L^p(\Omega; H)$ is closable.

Pf: If $(X_n) \subset \mathcal{S}$. st. $X_n \xrightarrow{a.s.} 0$. $DX_n \xrightarrow{L^p} \xi$.

$\mathbb{E}\,Y\langle \xi, h\rangle = \lim_{n\to\infty} \mathbb{E}(Y\langle DX_n, h\rangle)$

$= \lim_{n\to\infty} \mathbb{E}(YX_n W(h) - X_n\langle DY, h\rangle) = 0$. $\forall Y \in \mathcal{S}_b$. $h \in H$.

Def: Denote D is closure extension of D on $L^p(\Omega)$, with Domain $\mathbb{D}^{1,p}$

Rmk: i) $\mathbb{D}^{1,p}$ is closure of $\mathcal{S}$ under norm:

$$\|X\|_{1,p} := \left(\mathbb{E}|X|^p + \mathbb{E}\|DX\|_H^p \right)^{1/p}.$$

ii) $\mathbb{D}^{1,p}$ is reflexive since it's isometric to CLS of $L^p(\Omega) \times L^p(\Omega; H)$.

Prop. (Chain Rule)

For $p \geq 1$, $X = (X_1 \cdots X_m)$, $X_k \in \mathbb{D}^{1,p}$, and $\varphi \in C^1(\mathbb{R}^m)$, s.t. $|\nabla \varphi|$ is bdd. Then:
$\varphi(X) \in \mathbb{D}^{1,p}$ and $D\varphi(X) = \sum_1^m \partial_j \varphi(X) DX_j$.

Pf: Find $(X_k^{(n)}) \subset \mathcal{S} \xrightarrow{\mathbb{D}^{1,p}} X_k$.

$(\varphi_n) \subset C^\infty \xrightarrow{u.c.c} \varphi$.

By DCT, we have: $D\varphi_n(X^{(n)}) =$
$\sum \partial_j \varphi_n(X^{(n)}) DX_j^{(n)} \xrightarrow{L^p} \sum \partial_j \varphi(X) DX_j$

with $\varphi_n(X^{(n)}) \xrightarrow{L^p} \varphi(X)$.

So: $\varphi(X) \in \mathbb{D}^{1,p}$ from its closeness.

Next, we will prove general one for φ Lipschitz.

Lemma For $p>1$. $X_n \in \mathbb{D}^{1,p}$. s.t. $X_n \xrightarrow{L^p} X$. and

$\sup_n \mathbb{E}\, \|DX_n\|_H^p < \infty$. Then $X \in \mathbb{D}^{1,p}$. and

$DX_n \to DX$. weakly. in $\mathbb{D}^{1,p}$.

Pf: By reflexive: $\exists X_{n_k} \to Y \in \mathbb{D}^{1,p}$ in $\mathbb{D}^{1,p}$.

So: $X_{n_k} \to Y$ in L^p. $\Rightarrow Y = X \in \mathbb{D}^{1,p}$.

Besides. $\forall$ convergent subseq of DX_n

$\to DX$. $\Rightarrow DX_n \to DX$. in $\mathbb{D}^{1,p}$.

prop. For $p>1$. $X=(X_1 \cdots X_n)$. s.t. $X_j \in \mathbb{D}^{1,p}$. φ is

Lipschitz conti. with const. $= L$. Then:

$\varphi(X) \in \mathbb{D}^{1,p}$ and $\exists\, Y=(Y_1 \cdots Y_n)$. $|Y| \le L$.

s.t. $D\varphi(X) = \sum_1^n Y_i DX_i$ in weak sense.

Pf: Find $\varphi_n \in C^\infty \to \varphi$. pointwise. $|\nabla \varphi_n| \le L$.

By prop. above: $D\varphi_n(X) = \sum \partial_j \varphi_n(X) X_j$.

By Lemma: $D\varphi_n(X) \xrightarrow{\mathbb{D}^{1,p}} D\varphi(X)$.

By reflexive: $\exists n_k$. $\partial_j \varphi_{n_k}(X) \xrightarrow[(p>1)]{L^p} Y_j \in L^p(\Omega)$.

③ **High order:**

Next, we will define high order Malliavin deri.:

Def: For $X = \sum_{1}^{m} Y_j v_j \in \mathcal{S}(V)$, $Y_j = f_j(W(h_1^j), \cdots W(h_{n_j}^j)) \in \mathcal{S}$, $f_j \in C_p^\infty(\mathbb{R}^{n_j})$.

DX is a $H \otimes V$ r.v. (or $\mathcal{L}_{HS}(H, V)$-valued r.v.) defined by:

$$DX := \sum_i \sum_j \partial_i f_j(W(h_1^j) \cdots W(h_{n_j}^j))\, h_i^j \otimes v_j$$

$$= \sum_j (DY_j) \otimes v_j$$

Rmk: i) When $V = \mathbb{R}^1$, it coincides the def of Malliavin derivative before.

ii) It's indepdt of choice of (v_j) in representation of X.

check as before: $v_j = \sum \langle e_k, v_j \rangle e_k \ldots$

Lemma. $D: L^p(\Omega; V) \to L^p(\Omega; H \otimes V)$ is closable for $\forall p \geq 1$.

Pf: Set (v_j) is o.n.b of V.

$X_n = \sum Y_j^n v_j \xrightarrow{L^p} 0$.

$DX_n = \sum (DY_j^n) \otimes v_j \xrightarrow{L^p} \xi$.

$\Rightarrow Y_j^n \to 0$. $\forall j$. So: $DY_j^n \to 0$. $\forall j$.

Then by DCT. $DX_n \to 0$ in L^p.

Rmk: We define domain of closure of D by $\mathbb{D}^{1,p}(V)$.

Def: For $X \in \mathbb{D}^{1,p}$, $DX \in \mathbb{D}^{1,p}(H)$. We define D^2X by $D(DX)$, which's $H \otimes H$ valued r.v. (or $\mathcal{L}_{HS}(H)$-r.v.)

Inductively $\mathbb{D}^{k,p} := \{ X \in \mathbb{D}^{k-1,p}, D^{k-1}X \in \mathbb{D}^{1,p}(H^{\otimes(k-1)}) \}$ and $D^kX := D(D^{k-1}X)$.

Rmk: $\mathbb{D}^{k,p}$ is Banach space with norm

$$\|X\|_{k,p} := \left(E|X|^p + \sum_{j=1}^{k} \mathbb{E}\|D^jX\|^p_{H^{\otimes j}} \right)^{\frac{1}{p}}$$

prop. (Characterization of $\mathbb{D}^{1,2}$).

$J_n : L^2(\Omega, \mathcal{F}_W) \to \mathcal{H}_n$, orthogonal projection.

For $X \in L^2(\Omega)$, $X = \sum_{n\geq 0} J_n X$, chaos decomp.

Then: $X \in \mathbb{D}^{1,2} \iff \sum_{n\geq 0} n\|J_nX\|^2 < \infty$.

Besides, $DJ_nX = J_{n-1}DX$ and $\mathbb{E}\|DX\|_H^2 = \sum_{n\geq 0} n\|J_nX\|^2$.

Pf: Consider basis $(\phi_\alpha)_{|\alpha|=n}$ of $\mathcal{H}_n$.

$$D\phi_\alpha = \sum_l \prod_{j\neq l} H_{\alpha_j}(W(e_j))\, H_{\alpha_l - 1}(W(e_l)) \cdot \sqrt{\alpha!}$$

follows from $H_n' = H_{n-1}$.

$$\Rightarrow D\phi_\alpha = \sum_l \sqrt{\alpha_l}\, \phi_{\beta^l}, \quad \beta^j = (\alpha_1, \dots, \alpha_j - 1, \dots)$$

$\Rightarrow D\Phi_\alpha \in \mathcal{H}_{n-1}(H)$. and we have:

$\mathbb{E}\|D\Phi_\alpha\|^2 = \sum \alpha_i = |\alpha| = n$.

So: $\mathcal{H}_n \subset \mathbb{D}^{1,2}$. $D\mathcal{H}_n \subset \mathcal{H}_{n-1}(H)$

Besides. $\mathbb{E}\|DY\|^2 = n\,\mathbb{E}|Y|^2$. $\forall Y \in \mathcal{H}_n$.

We can obtain the conclusion by it!

<u>Cor.</u> If $X \in \mathbb{D}^{1,2}$. $DX = 0$. a.s. Then:

$X \equiv$ Const. a.s.

<u>Pf</u>: By prop. above: $J_n X = 0$. $\forall n \geq 1$

So: $X = J_0 X$. a.s.

<u>Cor.</u> $A \in \mathcal{F}_W$. Then $I_A \in \mathbb{D}^{1,2} \Leftrightarrow \mathbb{P}(A) \in \{0,1\}$.

<u>Pf</u>: $(\Leftarrow)$ It's trivial. $I_A \equiv$ const. a.s.

$(\Rightarrow)$ Find $\varphi \in C_c^\infty$. $\varphi = t^2$ on $(-2,2)$

$\Rightarrow D\varphi(I_A) = DI_A$

$= 2I_A \cdot DI_A$.

follows from chain rule.

So: $DI_A = 0$. Then $I_A \equiv$ Const.

(2) <u>Divergence Operator</u>:

Next, we consider adjoint. op. of D:

① **Def:** Divergence operator is adjoint δ of D on $L^2(\Omega; H)$. with $D(\delta) := \{ u \in L^2(\Omega; H) \mid \exists X \in L^2(\Omega).\ s.t.\ \mathbb{E}\langle DY, u\rangle_H = \mathbb{E}(YX).\ \forall Y \in \mathbb{D}^{1,2} \}$ and set $\delta(u) := X$.

Rmk: Note $\mathbb{D}^{1,2} \supset \mathcal{S}$ is dense in $L^2(\Omega)$.

So there's at most one X, a.s.

Lemma¹ $u \in L^2(\Omega; H)$. $u \in D(\delta) \Leftrightarrow \exists c \geq 0.\ st.$

$|\mathbb{E}\langle DY, u\rangle| \leq c\, \mathbb{E}^{\frac{1}{2}} |Y|^2$. for $\forall Y \in \mathbb{D}^{1,2}$.

Pf: By Riesz represent on $\mathbb{D}^{1,2}$.

Lemma² $\mathcal{S}(H) < D(\delta)$. So: $D(\delta)$ is dense in $L^2(\Omega, H)$

Besides, for $u = \sum_1^n X_i h_i$. we have:

$\delta(u) = \sum_1^n X_i W(h_i) - \langle DX_i, h_i\rangle$.

Pf: Use integrate-by-part formula and approxi. $\mathbb{D}^{1,2}$ by $\mathcal{S}$.

Lemma³ $[D_h, \delta](u) = \langle u, h\rangle$. for $\forall u \in D(\delta)$. $h \in H$.

where $D_h X = \langle DX, h\rangle$. for $X \in \mathbb{D}^{1,2}$. and

$D_h u = Du(h)$. $(Du \in \mathcal{L}_{HS}(H))$.

Pf: WLOG. set $u \in \mathcal{S}(H)$. and $u = Xg$

$= f(W(e_1) \cdots W(e_n))g$. $g, h \in \mathrm{span}\{e_i\}_1^n$. $e_i \perp e_j$.

By Lemma above. $\delta u = X W(g) - \langle DX, g\rangle$.

We can check $\langle D\delta(u), h\rangle - \delta(D u(h)) = \langle u, h\rangle$ directly by expand g, h into $(e_k)_1^n$.

<u>prop.</u> $\mathbb{D}^{1,2}(H) \subset D(\delta)$. Besides, for $u, v \in \mathbb{D}^{1,2}(H)$. We have $\mathbb{E}(\delta(u)\delta(v)) = \mathbb{E}\langle u, v\rangle + \mathbb{E}(\operatorname{tr}(Du\, Dv))$.

<u>Pf:</u> <u>Lemma 4</u> $D\langle u, h\rangle = Du(h)$.

<u>Pf:</u> $u = \sum_1^n X_i h_i$. (h_i) is o.n.b. $\in \mathcal{S}(H)$. easy to check.

Then use approximation.

Next, we assume $u, v \in \mathcal{S}(H)$ first. and set (e_k) is o.n.b of H.

Note: LHS $= \sum \mathbb{E}\langle u, e_k\rangle\langle D\delta(v), e_k\rangle$.

Then apply Lemma 3 and Lemma 4. to obtain the equation.

Finally. $\exists (u_n) \in \mathcal{S}(H) \xrightarrow{L^2} u$. and $Du_n \xrightarrow{L^2} Du$ for $\forall u \in \mathbb{D}^{1,2}(H)$.

So: $(\delta(u_n))$ is L^2-Cauchy since $\mathbb{E}|\delta(u_n)|^2 = \mathbb{E}\|u_n\|^2 + \mathbb{E}\|Du_n\|^2_{H\otimes H}$

$$\mathbb{E}(\langle DY, u\rangle) = \lim_{n\to\infty} \mathbb{E}(Y\delta(u_n)) \overset{\exists X \in L^2}{=} \mathbb{E}(YX)$$

combine with polarization we can obtain the result from $\mathbb{E}\|\delta(u)\|^2 = \cdots$

<u>prop.</u> (Scalar by $\mathbb{D}^{1,2}$.)

For $X \in \mathbb{D}^{1,2}$. $u \in D(\delta)$. st. $\begin{cases} Xu \in L^2(\Omega; H) \\ X\delta(u) - \langle DX, u\rangle \in L^2(\Omega) \end{cases}$

Then: $Xu \in D(\delta)$ and $\delta(Xu) = X\delta(u) - \langle DX, u\rangle$.

<u>Pf:</u> For $\tilde{X} \in \mathcal{S}$ and $Y \in \mathcal{S}_b$.

$$\mathbb{E}(\langle DY, \tilde{X}u\rangle) = \mathbb{E}(Y(\tilde{X}\delta(u) - \langle D\tilde{X}, u\rangle))$$

Then by density of $\mathcal{S}$ and $\mathcal{S}_b$.

<u>Def:</u> T on $(\Omega, \Sigma, \mathbb{P})$ is local if $\forall X = 0$ a.s. on $A \in \Sigma$. $\Rightarrow TX = 0$ a.s. on A.

<u>Rmk:</u> For vector space R. If $\xi \in R_{loc}$. i.e. $\exists (A_n, \xi_n) \subset \Sigma \times R$. st. $A_n \uparrow \Omega$. and $\xi = \xi_n$ on A_n. Then for $\forall$ Linear local operator T on R. we can def: $T\xi := T\xi_n$ on A_n. Since $T(\xi_n - \xi_m) = 0$ on A_m. $m \leq n$. it's well-def.

<u>prop.</u> i) δ is local in $\mathbb{D}^{1,2}(H)$

ii) D is local in $\mathbb{D}^{1,1}$

Rmk: We can extend δ, D to $\mathbb{D}^{1,2}_{loc}(H)$ and $\mathbb{D}^{1,p}_{loc}$ respectively.

(Note $\forall f \in \mathbb{D}^{1,p}_{loc}$, (A_n, fI_{A_n}) is the sequence. where $A_n \uparrow \Omega$.)

Pf: i) Prove: $\delta(u) \cdot I_{\{\|u\|=0\}} = 0$. a.s.

$\Leftrightarrow \mathbb{E}(X\delta(u) I_{\|u\|=0}) = 0$. $\forall X \in \mathcal{S}_c$.

Find $\varphi \in C_c^\infty(\mathbb{R}')$. s.t. $I_{(-1,1)} \le \varphi \le I_{(-2,2)}$

Set $\varphi_n(t) = \varphi(nt) \to \delta$. and also:

$\varphi_n'(t) \to \infty \cdot \delta$.

Consider: $\mathbb{E}\langle u, D(X\varphi_n(\|u\|^2))\rangle = \cdots$

Let $n \to \infty$ with DCT.

ii) WLOG. Set $X \in \mathbb{D}^{1,1}$ and $X \in L^\infty$.

(Otherwise replace X by $\arctan X$).

Set $\psi_n(t) := \int_{-\infty}^t \varphi_n$. φ_n is func. above.

Note: $|\mathbb{E}\langle D\psi_n(X), u\rangle| = |\mathbb{E}\,\varphi_n(X)\langle DX, u\rangle|$

$\le \frac{1}{n}\mathbb{E}|\delta(u)| \cdot \|\varphi\|_1 \to 0$ $\forall u \in \mathcal{S}_b$

So: $\mathbb{E}(I_{\{\|X\|=0\}}\langle DX, u\rangle) = 0 \Rightarrow I_{\|X\|=0}DX \overset{a.s.}{=} 0$

(3) Ornstein-Uhlenbeck Semigroup:

Def: O-U semigroup $(T(t))_{t\ge0}$ is defined by:

$$T(t)X = \sum_{n\ge0} e^{-nt} J_n X \text{. for } X \in L^2(\Omega, \Sigma_W, \mathbb{P}).$$

<u>Lemma.</u> i) $T(t) \in \mathcal{L}(L^2(\Lambda))$ and it's self-adjoint.

ii) $(T(t))_{t \geq 0} \in C_0$ on $L^2(\Lambda)$.

<u>prop.</u> The generator L of $(T(t))_{t \geq 0}$ is given by

$LX := \sum_{n \geq 0} -n J_n X$. $\forall X \in D(L) := \{ X \in L^2(\Lambda) \mid \sum n^2 \| J_n X \|_2^2 < \infty \}$.

<u>Pf:</u> Check: O-U operator $AX := \lim_{t \to 0} \frac{T(t)X - X}{t}$ coincides LX. Note: $[J_n, T(t)] = 0$.

1) If $X \in D(A)$, $AX = Y$. Then:

$$J_n Y = \lim_{t \to 0} \frac{T(t) J_n X - J_n X}{t} = -n J_n X.$$

2) If $X \in D(L)$. $\mathbb{E} \left| \frac{T_t X - X}{t} - LX \right|^2 \to 0$.

<u>prop.</u> $L = -\delta D$. i.e. $D(L) = \{ X \in \mathbb{D}^{1,2} : DX \in D(\delta) \}$.

<u>Pf:</u> 1) $\forall X \in \mathbb{D}^{1,2}$, $DX \in D(\delta)$. $\forall Y \in \mathbb{D}^{1,2}$.

$$\mathbb{E}(Y \delta D X) = \mathbb{E} \langle DY, DX \rangle$$
$$= \sum_{n \geq 0} \mathbb{E} \langle J_n DY, J_n DX \rangle$$
$$= \sum_{n \geq 0} n \, \mathbb{E}(J_n Y J_n X) = \mathbb{E} Y(-LX)$$

$\Rightarrow$ Set $Y \in \mathcal{H}_m$. Then: we have,

$$\mathbb{E}(Y J_m(\delta D X - m J_m X)) = 0.$$

It also holds for $Y \in \mathcal{H}_m^\perp$. So:

$$J_m(\delta D X) \overset{a.s.}{=} m J_m X \Rightarrow \sum_{n \geq 0} n^2 \| J_n X \|_2^2 < \infty$$

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2) $\forall X \in D(L) \Rightarrow X \in \mathbb{D}^{1,2}$.

Besides. $\mathbb{E}\langle DY, DX\rangle = \mathbb{E}(Y(-LX))$

So: $DX \in D(\delta)$ and $\delta DX = -LX$.

Thm. (Hypercontractivity)

$\forall p, q > 1$. $t \geq 0$. s.t. $\frac{p-1}{q-1} = e^{2t}$. Then:

$\|T_t X\|_{L^p} \leq \|X\|_{L^2}$. $\forall X \in L^2(\Omega, \Sigma_W, \mathbb{P})$

Cor. (Reverse Jensen's inequi.)

$\forall X \in \mathcal{H}_n$. $p \geq 1$. $\mathbb{E} X^{2p} \leq (2p-1)^{np} (\mathbb{E} X^2)^p$

Rmk: An important feature is that it holds without appearance of any const. which makes it have tensor-property.

Cor. (Tensorization property)

$T_i : L^q(\Omega_i, \mathbb{P}_i) \xrightarrow{BLO} L^p(\Omega_i, \mathbb{P}_i)$. $i = 1, 2$.

If we set $\Omega = \Omega_1 \times \Omega_2$. $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$.

$T = T_1 \otimes T_2$. Then: $\|TX\|_p \leq \|X\|_2$.

Pf: Lemma. If $p \geq q \geq 1$. Then, we have:

$$\| \|X\|_{L^q(\Omega_1)} \|_{L^p(\Omega_2)} \leq \| \|X\|_{L^p_2} \|_{L^q_1}$$

Pf: First prove when $q = 1$.

Note: $\|X\|_2 = \| |X|^2 \|_1^{1/2}$.