Stochastic Calculus

(1) Multiple Wiener integral:

Next. we fix $H = L^2(T, \mathcal{B}, m)$. where $(T, \mathcal{B}, m)$ is σ-finite, m is nonatomic.

Def: i) $\mathcal{B}_0 = \{ A \in \mathcal{B} \mid m(A) < \infty \}$.

ii) $m \in \mathbb{Z}^{\geq 1}$. the space of elementary process

$$\mathcal{E}_m := \{ f \in L^2(T^m, \mathcal{B}^m, \otimes^m m) \mid f = \sum_1^n a_{i_1 \cdots i_m} I_{A_{i_1} \times \cdots A_{i_m}}. \ (A_k)_1^n \text{ pairwise disjoint} \subset \mathcal{B}_0. \ a_{i_1 \cdots i_m} = 0 \text{ if } \exists i_k = i_j. \}$$

Rmk: $\mathcal{E}_m$ is generated by $I_{B_1 \times \cdots \times B_m}$ where $B = B_1 \times \cdots B_m \cap \Delta_{ij} (\stackrel{\Delta}{=} \{t_i = t_j\}) = \emptyset$

Lemma. $\forall m \geq 1$. $\mathcal{E}_m$ is dense in $L^2(T^m, \mathcal{B}^m, m^{\otimes m})$.

Pf: prove: $I_{A_1 \times \cdots A_m} \in \overline{\mathcal{E}_m}$. where $A_i \in \mathcal{B}_0$.

For $\varepsilon > 0$. find (E_i) disjoint pairwise. $m(E_i) \leq \varepsilon$. and $\bigcup E_i = \bigcup A_i$.

So $\exists (\varepsilon_{i_1 \cdots i_m}) \subset \{0, 1\}$. st.

$$I_{A_1 \cdots \times A_m} = \sum_1^n \varepsilon_{i_1 \cdots i_m} I_{E_{i_1} \times \cdots \times E_{i_m}}$$

Set $$I_B = \sum_I \varepsilon_{i_1 \cdots i_m} I_{E_{i_1} \times \cdots \times E_{i_m}}.$$

$I = \{ (i_1 \cdots i_m) \mid \forall i_j \neq i_k. \}$.

KOKUYO

$$\delta_1 : \| I_{A_1\times\cdots\times A_m} - I_B \|_{L^2} \overset{(*)}{\leq}$$

$$\binom{m}{2} \sum m(E_k)^2 \left(\sum m(E_j)\right)^{m-2} \lesssim \varepsilon.$$

$(*)$: $\binom{m}{2}$ possibilities fix a pair.

<u>Rmk</u>: It's not true if m is atomic.

<u>e.g.</u> $T = \{0,1\}$. $m(\{1\}) = m(\{0\}) = 1$.

<u>Def</u>: m-fold Wiener integral $I_m(f)$ defined by:

$$I_m(f) := \sum_1^n a_{i_1\cdots i_m} W(A_{i_1}) \cdots W(A_{i_m}) \text{ for}$$

$$f = \sum a_{i_1\cdots i_m} I_{A_{i_1}\times\cdots\times A_{i_m}} \in \mathcal{E}_m$$

<u>Rmk</u>: I_m is linear operator.

<u>Lemma.</u> For $f \in \mathcal{E}_m$. $g \in \mathcal{E}_k$.

i) $\tilde{f}(t_1 \cdots t_m) := \frac{1}{m!} \sum_{\sigma \in S_m} f(t_{\sigma(1)} \cdots t_{\sigma(m)})$

symmetrization of f. Then: $I_m(f) = I_m(\tilde{f})$.

ii) $$\mathbb{E}(I_m(f) I_k(g)) = \begin{cases} 0 & . \quad k \neq m \\ m! \langle \tilde{f}, \tilde{g} \rangle_{L^2} & . \quad k = m. \end{cases}$$

<u>Pf</u>: i) trivial: $W(A_1) \cdots W(A_m) = W(A_{\sigma(1)}) \cdots W(A_{\sigma(m)})$

ii) Consider f, g are under the same partition (A_k). and set $f = \tilde{f}$.

$g = \tilde{g}$. easy to check.

Rmk: Note $\| I_m(f) \|_{L^2} = \| I_m(\tilde{f}) \|_{L^2}$

$= m! \| \tilde{f} \|_{L^2} \le m! \| f \|_{L^2}$

$\Rightarrow I_m$ is BLO from $\mathcal{E}_m$ to $L^2(\Omega)$.

By density of $\mathcal{E}_m$. extend I_m:

from $L^2(T^m)$ to $L^2(\Omega)$.

Denote $I_m(f) := \int_{T^m} f(t_1 \cdots t_m)\, dW(t_1) \cdots dW(t_m)$

Denote: For $f \in L^2(T^m)$, $g \in L^2(T^k)$.

i) $f \otimes g(t_1 \cdots t_{m+k}) := f(t_1 \cdots t_m)\, g(t_{m+1} \cdots t_{m+k})$

ii) $f \otimes_1 g(t_1 \cdots t_{m+k-2}) := \int_T f(t_1 \cdots t_{m-1}, s)\, g(t_m, \cdots t_{m+k-2}, s)\, d\mu(s)$

prop. For $\tilde{f} \in L^2(T^m)$. sym. and $g \in L^2(T^1)$. Then:

$I_m(\tilde{f}) I_1(g) = I_{m+1}(\tilde{f} \otimes g) + m I_{m-1}(\tilde{f} \otimes_1 g)$.

Rmk: Symmetric is required:

Note $I_{m-1}(f \otimes_1 g) \neq I_{m-1}(\tilde{f} \otimes_1 g)$

Pf: WLOG. assume $\tilde{f} = \tilde{I}_{A_1 \times \cdots A_m}$. $g = I_B$.

(A_k) pairwise disjoint. $\in \mathcal{B}_0$.

i) $B \cap A_k = \emptyset$. $\forall k$. $\Rightarrow \tilde{f} \otimes_1 g = 0$. trivial.

ii) $B \cap A_k \neq \emptyset$. $\exists k$. WLOG. set $B = A_1$.

Check: $f \otimes_1 g = \frac{1}{m} \tilde{I}_{A_2 \times \cdots A_m}\, \mu(A_1)$.

Find (B_j). s.t. $A_1 = \overset{\sim}{\underset{i}{\cup}} B_j$. $m(B_j) \leq \varepsilon$.

and B_j are disjoint, measurable

$$I_m(\tilde{f}) I_1(g) = \left(\sum_i^{\sim} W(B_j)\right)^2 W(A_2) \cdots W(A_m)$$

$$= m(A_1) W(A_2) \cdots W(A_m) + \sum_{i \neq j} W(B_i) W(B_j) W(A_2) \cdots$$

$$+ \sum (W(B_i)^2 - m(B_i)) W(A_2) \cdots W(A_m)$$

$$= m I_{m-1}(\widetilde{f \otimes_1 g}) + I_{m+1}(h_\varepsilon) + R_\varepsilon.$$

where $h_\varepsilon = \sum_{i \neq j} I_{B_i \times B_j \times A_2 \times \cdots \times A_m}$.

Note: $\mathbb{E}(R_\varepsilon^2) \leq 2 \sum m(B_j)^2 m(A_2) \cdots \leq C\varepsilon$.

$$\|\tilde{h}_\varepsilon - \widetilde{f \otimes g}\|_{L^2} \leq \|h_\varepsilon - I_{A_1 \times A_1 \times A_2 \cdots}\|$$

$$\leq C\varepsilon.$$

prop. For $h \in L^2(T)$. $\|h\| = 1$. Then:

$$m!\, H_m(W(h)) = \int_{T^m} h(t_1) \cdots h(t_m)\, dW(t_1) \cdots dW(t_m)$$

Besides, $I_m : L^2(T^m) \to \mathcal{H}_m$. Surjective and

$I_m(f) = I_m(g) \iff \tilde{f} = \tilde{g}$.

Pf: 1) Process by induction. Note RHS $= I_m(h^{\otimes m})$.

use prop. above to transit $m+1 \to m$.

and $(m+1) H_{m+1}(x) = x H_m(x) - H_{m-1}(x)$.

2) $L^2_{sym}(T^m) = \{\tilde{f} \mid f \in L^2(T^m)\}$.

Note: $\mathbb{E}(I_m(\tilde{f})^2) = m!\, \|\tilde{f}\|_{L^2(T^m)}$

So: $I_m(L^2_{sym}(T^m))$ is close in $L^2(\Omega)$

and $I_m(\tilde{f}-\tilde{g})=0 \iff \tilde{f}=\tilde{g}$ a.s.

$\Rightarrow \mathcal{H}_m \subset I_m(L^2_{sym}(T^m))$.

But $I_m(L^2_{sym}) \perp \mathcal{H}_k$. $\forall k \neq m$. $(I_k \perp I_m)$

So: $I_m(L^2_{sym}) = \mathcal{H}_m$. $I_m : L^2_{sym} \to \mathcal{H}_m$ biject.

<u>Cor.</u> $I_m(\bigotimes_1^r h_i^{\otimes k_i}) = k! \prod_1^r \mathcal{H}_{k_i}(W(h_i))$.

where $\sum_1^r k_i = m$. $k! = k_1! \cdots k_r!$

<u>Cor.</u> $\forall X \in L^2(\Omega)$. $\exists f_n \in L^2_{sym}(T^n)$. unique.

St. $X = \sum_{n \geq 0} I_n(f_n)$. $f_0 = \mathbb{E}(X)$. $I_0 = id$.

<u>Thm</u> (Fourth Moment Thm)

For $n \geq 2$. $(F_k) \subset \mathcal{H}_n$. St. $\mathbb{E}(F_k^2) = 1$. $\forall k \geq 1$.

Then: $\mathbb{E}(F_k^4) \xrightarrow{k\to\infty} 3 \iff F_k \xrightarrow[k\to\infty]{d} N(0,1)$

<u>Lemma.</u> For $n \geq 2$. $F \in \mathcal{H}_n$. $F \neq 0$. Then $\exists c = c(n)$

St. $\mathbb{E}(F^4) - 3(\mathbb{E}(F^2))^2 \geq c(\mathbb{E}\|DF\|^4 - \mathbb{E}^2(\|DF\|))$

> 0.

<u>Rmk:</u> It means $\mathcal{H}_n$ itself can't contain any Gaussian r.v. !

(2) <u>Calculus for White Noise:</u>

Note that when $H = L^2(T, \mathcal{B}, m)$, $X \in \mathbb{D}^{1,2}$

$\Rightarrow DX \in L^2(\Omega; H)$. We can see $L^2(\Omega; H)$

$= L^2(\Omega \times T)$. Similar for $D^k X \in L^2(\Omega \times T^k)$

e.g. $D_s D_t X = \sum_{i,j}^{n} \partial_i \partial_j f(W(h_1) \cdots W(h_n)) h_i(s) h_j(t)$.

for $X = f(W(h_1) \cdots W(h_n))$.

Prop. $X \in \mathbb{D}^{1,2}$, $X = \sum_{n \geq 0} I_n(f_n)$, $f_n \in L^2_{sym}(T^n)$.

Then: $D_t X = \sum_{n \geq 1} n I_{n-1}(f_n(\cdot, t))$.

Rmk: It means derivative of X obtained by removing one of integral.

Pf: WLOG. Set $X = I_m(f_m)$, where

$f_m = \sum a_{i_1 \cdots i_m} I_{A_{i_1} \times \cdots A_{i_m}} \in \mathcal{E}_{m, sym}$.

check it directly.

Def: For $A \in \mathcal{B}$, set $\Sigma_A := \sigma(W(B) \mid B \in \mathcal{B}_0, B \subset A)$, σ-algebra w.r.t. A.

Lemma. $X = \sum I_n(f_n) \in L^2(\Omega)$, $A \in \mathcal{B}$. Then:

$\mathbb{E}(X \mid \Sigma_A) = \sum_{n \geq 0} I_n(f_n I_A^{\otimes n})$

Pf: Set $f = I_n(f_n)$. $f_n = I_{B_1 \times \cdots B_n}$. where

B_i pairwise disjoint. finite measure.

prop. If $X \in \mathbb{D}^{1,2}$. $A \in \mathcal{B}$. Then $\mathbb{E}(X \mid \mathcal{I}_A) \in \mathbb{D}^{1,2}$

and $D_t \mathbb{E}(X \mid \mathcal{I}_A) = \mathbb{E}(D_t X \mid \mathcal{I}_A) I_A$.

Pf: It's direct from Lemma and prop above.

Cor. $A \in \mathcal{B}$, $X \in \mathbb{D}^{1,2}$: $\mathcal{I}_A$-measurable $\Rightarrow D_t X = 0$

a.s on $\Omega \times A^c$.

e.g. $T = \mathbb{R}^{\geq 0}$. $\mathcal{G}_t = \sigma(B_s, s \leq t)$. Then

$\text{supp}\, D_s X \subset [0, t]$. if $X \in \mathcal{G}_t$ (e.g. $X = B_t$).

Def: i) We set Skorokhod integral of $u(w,t) \in D(\delta)$

$\subset L^2(\Omega \times T)$. by $\delta(u) := \int_T u(t)\, \delta W(t)$.

ii) For $f_n(t_1, t_2 \cdots t_n, t)$ sym in the first n var.

set $\widetilde{f_n} = \frac{1}{n+1}\left(f(t_1 \cdots t_n, t) + \sum_j f(t_1 \cdots t, t_{j+1} \cdots t_j) \right)$

prop. $u = \sum_{n \geq 0} I_n(f_n(\cdot, t)) \in L^2(\Omega \times T)$. Then $u \in D(\delta)$

$\Leftrightarrow \sum_{n \geq 0} (n+1)! \, \|\widetilde{f_n}\|^2_{L^2(T^{n+1})} < \infty$. In this case:

Besides. $\delta(u) = \sum_{n \geq 0} I_{n+1}(\widetilde{f_n})$.

Rmk. Contrary to D. δ add an integral.

Pf: For $Y = I_n(g)$. g is symmetric.

Check: $\mathbb{E}\langle u, DY\rangle = \mathbb{E}(I_n(\tilde{f}_{n-1})g)$

($\Rightarrow$). So: $J_n\delta(u) = I_n(\tilde{f}_{n-1})$.

$\delta(u) = \sum_{n\geq 0} I_{n+1}(\tilde{f}_n)$. We have:

$\mathbb{E}(\delta(u)^2) = \sum_{n\geq 0}(n+1)!\,\|\tilde{f}_n\|^2_{L^2(T^{n+1})} < \infty$.

($\Leftarrow$) $Z := \sum I_{n+1}(\tilde{f}_n)$ exists.

Besides: $\mathbb{E}\langle u, DY\rangle = \mathbb{E}\,ZY$.

for $\forall Y \in \mathcal{H}_n$. $\Rightarrow \forall Y \in \bigotimes \mathcal{H}_n$.

Lemma $A \in \mathcal{B}_0$. $X \in L^2(\Omega)$. $\mathcal{F}_{A^c}$-measurable $\Rightarrow X I_A \in D(\delta)$.

and $\delta(X I_A) = X W(A)$.

Rmk: $\mathbb{D}^{1,2}(H) \subsetneq D(\delta)$.

since for $\mu(A) > 0$. $X I_A \in \mathbb{D}^{1,2}(H)$

$\Leftrightarrow X \in \mathbb{D}^{1,2}$. We set $X = I_{\{W(B)>0\}}$.

$B \cap A = \emptyset$. So: $\mathbb{P}(W(B) > 0) = \frac{1}{2} \notin \{0,1\}$.

$X \notin \mathbb{D}^{1,2}$. $\Rightarrow X I_A \notin \mathbb{D}^{1,2}(H)$.

Pf: Assume $X \in \mathbb{D}^{1,2}$. $\Rightarrow DX = 0$. a.e. in $\Lambda \times A$.

By scalar prop: $X I_A \in D(\delta)$ and:

$\delta(X I_A) = X W(A) - \langle DX, I_A\rangle = X W(A)$.

For $X \in L^2(\Omega)$, find $(X_n) \subseteq \mathbb{D}^{1,2} \xrightarrow{L^2} X$

and use closeness of $\delta = D^*$.

<u>prop.</u> (Commutative relation)

$u \in \mathbb{D}^{1,2}(H)$, s.t. $\forall t$, $\omega \mapsto D_t u(\omega) \in D(\delta)$ and

$t \mapsto \delta D_t u(\omega) \in L^2(\Omega \times T)$ a.s. Then $\delta(u) \in \mathbb{D}^{1,2}$.

and $[D_t, \delta](u) = u(t)$

<u>Pf:</u> 1) $u = \sum I_n(f_n(\cdot, t))$, $D_s u = \sum n I_{n-1}(f_n(\cdot, s, t))$

$\delta D_s u = \sum n I_n(\widetilde{f_n}(\cdot, t))$ All $\in L^2$

$\Rightarrow \sum n^2 \cdot n! \|\widetilde{f_n}\|^2_{L(T^{n+1})} < \infty$

Note $J_n(\delta(u)) = I_n(\widetilde{f_{n-1}})$

So, $\sum n \|J_n(\delta(u))\|^2 < \infty$. $\delta(u) \in \mathbb{D}^{1,2}$

2) It's direct to check $[D_t, \delta](\cdot) = (\cdot)_t$.

(3) <u>Itô integral:</u>

<u>Denote:</u> $\mathbb{L}^{1,2} := \mathbb{D}^{1,2}(H) \subset D(\delta)$. Hilbert space with

$\|u\|^2_{\mathbb{L}^{1,2}} := \|Du\|^2_{L^2(\Omega \times T^2)} + \|u\|^2_{L^2(\Omega \times T)}$

For $H = L^2(T, \mathcal{B}, \mu)$, set $T = \mathbb{R}^{\geq 0}$, $\mathcal{B} = \mathcal{B}_{\mathbb{R}^{\geq 0}}$, $\mu = dt$.

$W(I_{[0,t)}) = B_t$. $\mathbb{F} := \sigma(B_s, s \geq 0)$. $\mathcal{F}_t = \sigma(B_s, s \leq t)$.

For $u(t)$ is adapted, we have Itô isometry:

$$\mathbb{E}(\delta(u)^2) = \mathbb{E}\left(\int_0^\infty u^2(t)\,dt\right).$$

Pf: Recall: $\mathbb{E}(\delta(u)^2) = \mathbb{E}\int_0^\infty u^2\,dt + \mathbb{E}\left(\int_{[0,\infty)^2} D_s u(t) D_t u(s)\right)$

But $D_s u(t) D_t u(s) = 0$ if $t \neq s$.

Def: i) Elementary step process $u = \sum_1^n X_j I_{(t_j, t_{j+1}]}$

where $X_j \in L^2(\Omega, \mathcal{F}_{t_j}, \mathbb{P})$

Rmk: By Lemma before:

$$\delta(u) = \sum_1^n X_j (B_{t_{j+1}} - B_{t_j})$$

prop. $L^2_{\mathbb{F}}(\Omega \times \mathbb{R}^{\geq 0}) \subset D(\delta)$.

Pf: $\forall f \in L^2_{\mathbb{F}}$. $\exists f_n \xrightarrow{L^2} f$. where

(f_n) is seq of step functions.

By Itô isometry. $(\delta(f_n))$ is

Cauchy converges in L^2.

$\Rightarrow f \in D(\delta)$ by closeness.

ii) Set $\delta(u) := \int_0^\infty u(t)\,dt$ for $u \in L^2_{\mathbb{F}}$.

Itô integral of u.

Rmk: The prop above show: Itô-integrable

$\Rightarrow$ Skorokhod-integrable. and that

Skorokhod integral coincides Itô integral.

<u>Prop.</u> $u \in L^2_{\mathbb{F}}(\Omega \times (0,2))$. $X := \int_0^2 u(s)\,dB_s$. Then:

$X \in \mathbb{D}^{1,2} \iff u \in \mathbb{L}^{1,2}$. Besides. $t \mapsto D_t u(s) \in L^2_{\mathbb{F}}$

and $D_t X = u(t) + \int_t^2 D_t u(s)\,dB_s$. a.s.

<u>Pf:</u> $(\Leftarrow)$. $D_t u(s) \in L^2 \Rightarrow D_t u(s) \in D(\delta)$. by prop.

Use: $D_t \delta(u) = u(t) + \delta D_t u$. $X = \delta(u)$.

$(\Rightarrow)$ Set: u_n is ortho-proj. of u on $\mathcal{P}_n$.

$$X_n := \int_0^2 u_n(t)\,dB(t)$$

So: X_n is ortho-proj. of X on $\mathcal{P}_{n+1}$

By Itô isometry and the formula above

$X_n \xrightarrow{\mathbb{D}^{1,2}} X$. So: $c \geq \mathbb{E}\int_0^2 |D_t X_n|^2$

$= \mathbb{E}\int_0^2 |u(t) + \int_t^2 D_t u_n(s)\,dB_s|^2 \geq \mathbb{E}\,\|Du_n\|_2^2$.

So $u \in \mathbb{D}^{1,2}(H)$ by the convergence lemma.

<u>Thm.</u> (Clark-Ocone Formula)

$(T, \mathcal{B}, \mu) = ([0,2], \mathcal{B}_{[0,2]}, dt)$. If $X \in \mathbb{D}^{1,2}$.

Then: $X = \mathbb{E}(X) + \int_0^2 \mathbb{E}(D_t X \mid \mathcal{F}_t)\,dB_t$

<u>Rmk:</u> By repre.: $X = \mathbb{E}(X) + \int_0^2 u(t)\,dt$. we have find $u(t)$ in this Thm.

<u>Pf:</u> Set $X = \sum I_n(f_n)$. $\Rightarrow \mathbb{E}(D_t X \mid \mathcal{F}_t) = \sum_{n \geq 1} n I_{n-1}(f\, I_{[0,t]}^{\otimes n-1})$

check $\delta(\mathbb{E}(D_t X \mid \mathcal{F}_t)) = X - \mathbb{E}(X)$.