Multivariate Holomorphic

(1) Holomorphic in $\mathbb{C}^n \to \mathbb{C}^1$:

① **Def**: i) For $a = (a_1 \cdots a_n) \in \mathbb{C}^n$. $r = (r_1 \cdots r_n) \in \mathbb{R}^n$.

$\Delta(a, r) =: \{ z \in \mathbb{C}^n \mid |z_i - a_i| < r_i,\ 1 \le i \le n \}$

$B(a, \tilde{r}) =: \{ z \in \mathbb{C}^n \mid \|z - a\|_2^2 < \tilde{r}^2 \}$. $\tilde{r} \in \mathbb{R}^1$.

$\Delta^n =: \Delta(0, 1)$. $B^n =: B(0, 1)$.

Rmk: By Poincare inequivalent Thm.

$B^n \not\cong \Delta^n$. $\forall n \ge 2$.

ii) For $\Omega \subseteq \mathbb{C}^n$. $A(\Omega) =: \{ f : \Omega \to \mathbb{C}^1 \mid$ $\forall a \in \Omega$. $\exists U(a)$ nbd of a. st. $f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (z - a)^\alpha$. $\}$ set of holomorphic on Ω

Rmk: i) Value of $\sum_{\alpha \in \mathbb{N}^n} a_\alpha$ won't depend on the order of (a_α) if it's absolutely convergent.

ii) $A(\mathbb{C}^\omega) = \emptyset$. Since there's no open nbd where the series converges.

iii) $dz_k =: dx_{2k-1} + i\, dx_{2k}$. $d\bar{z}_k =: dx_{2k-1} - i\, dx_{2k}$

$\frac{\partial}{\partial z_k} =: \frac{1}{2} \left(\frac{\partial}{\partial x_{2k-1}} + \frac{1}{i} \frac{\partial}{\partial x_{2k}} \right)$. $\frac{\partial}{\partial \bar{z}_k} =: \frac{1}{2} \left(\frac{\partial}{\partial x_{2k-1}} - \frac{1}{i} \frac{\partial}{\partial x_{2k}} \right)$

$$df =: \sum_{j=1}^{2n} \frac{\partial f}{\partial x_j} dx_j = \sum_1^n \frac{\partial f}{\partial z_j} dz_j + \sum_1^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

$$\triangleq \partial f + \bar{\partial} f.$$

<u>Lemma.</u> (Abel's)

i) If $(b_\alpha a^\tau)_{\alpha \in \mathbb{N}^n}$ is bdd. $a_i \neq 0$. $\forall i$.

Then. $\sum_{\alpha \in \mathbb{N}^n} b_\alpha z^\alpha$ will uniformly and absolutely converge on every cpt set of $\Delta(0, a)$. So does $\sum b_\alpha (\frac{\partial}{\partial z})^\beta z^\alpha$.

ii) $\Omega \subset \mathbb{C}^n$ is convergence domain for $\sum b_\alpha z^\alpha$. $\Rightarrow$ $\sum b_\alpha z^\alpha$ will unif and abs converges on every cpt set of Ω.

<u>Pf:</u> i) $\forall K \underset{cpt}{\subset} \Delta(0, a)$. $\exists\, r_0 < 1$. s.t.

$K \subset \Delta(0, r_0 a)$. Apply condition.

ii) By finite cover of K:

$K \subset \bigcup_k^n \Delta(0, r_0 a^{(k)})$. $a^{(k)} \in \Omega$.

<u>Rmk:</u> When differentiating the series in the $\mathbb{C}^n$. it's identical as the univariate case. Since we fix other z_k and $\frac{\partial}{\partial z_i}$ once per time.

Thm (Uniqueness)

$\Omega \subset \mathbb{C}^n$. domain. For $f \in A(\Omega)/\{0\}$.

$\Rightarrow f^{-1}\{0\}$ has no interior point.

Rmk: accumulation points won't necessarily be point of uniqueness in high dim. Actually, zeros are never isolated.

Pf: Note $\text{int} f^{-1}\{0\} \subset \{ f^{(\alpha)}(z) = 0.\ \forall \alpha \in \mathbb{N}^n \}$.

prove: $\{ f^{(\alpha)}(z) = 0.\ \forall \alpha \in \mathbb{N}^n \}$ is clopen.

i) $\{\cdots\} = \bigcap_{\alpha \in \mathbb{N}^n} \{ f^{(\alpha)}(z) = 0 \}$ closed.

2) For $a \in \{\cdots\}$ since $f(z) = \sum \frac{f^{(\alpha)}(a)}{\alpha!} (z-a)^\alpha$ holds in $U(a)$

$\Rightarrow U(a) \subset \{\cdots\}$.

By connectedness of $\Omega \Rightarrow \{\cdots\} = \Omega$.

Lemma. $\Omega \subset \mathbb{C}^n$. domain. $f: \Omega \to \mathbb{C}^1$ is conti / bdd.

If $\forall a \in \Omega$. $f(\hat{a}^j, z) \in A(\Omega_{j,a})$. where $(\hat{a}^j, z) = (a_1, \cdots a_{j-1}, z, a_{j+1} \cdots a_n) \in \mathbb{C}^n$. $\Omega_{j,a} =: \{ z \in \mathbb{C}^1 \mid (\hat{a}^j, z) \in \Omega \}$. Then, $f \in A(\Omega)$

Pf: Fix other variables each time:

$$f(z) = (2\pi i)^{-n} \int_{|\zeta_1 - p_1| = r_1} \frac{d\zeta_1}{\zeta_1 - z_1} \cdots \int_{|\zeta_n - p_n| = r_n} \frac{f(\zeta)\, d\zeta_n}{\zeta_n - z_n}$$

for $z \in \Delta(\vec{p}, \vec{r})$.

Since f is conti / bdd. So: (exchange $\int$)

$$f(z) = \int_{|\xi_1 - p_1| = r_1} \cdots \int_{|\xi_n - p_n| = r_n} \frac{f(\xi)\, d\xi}{\prod^n (\xi_i - z_i)} .$$

$$1/\xi_i - z_i = 1/((\xi_i - p_i) - (z_i - p_i)) = \sum_{\alpha_i \in \mathbb{N}^n} \frac{(z_i - p_i)^{\alpha_i}}{(\xi_i - p_i)^{\alpha_i + 1}}$$

$\Rightarrow$ Expand $f(z) = \sum_\alpha (2\pi i)^{-n} \int_\square \frac{f(\xi) d\xi}{\prod (\cdots)^{\alpha_i+1}} \cdot (z-p)^\alpha .$

Thm. (Characterization of $A(\Omega)$)

$A(\Omega) = \{ f \in C^1(\Omega) \mid \partial f / \partial \bar{z}_k = 0.\ \forall k \}.$

Pf: By Lemma above.

Thm. For $f \in A(\Delta(0, r))$. $\Rightarrow f(z) = \sum \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha$ on $\Delta(0, r)$.

Pf: Consider $\bar{\Delta}(0, \varepsilon) \subset \Delta(0, r)$.

expand f on $\bar{\Delta}(0, \varepsilon)$. Set $\varepsilon \to r$.

Thm. (Cauchy inequi.)

$\bar{\Delta}(a, r) \subset \Omega \subset \mathbb{C}^n$. If $f \in A(\Omega)$. Then:

$|f^{(\alpha)}(a)| \le \frac{\alpha!}{r^\alpha} \cdot \sup \{ |f(z)| \mid |z_i - a_i| = r_i.\ 1 \le i \le n \}.$

Pf: $LHS \le \frac{\alpha!}{(2\pi)^n} \int_\square \frac{|f(\xi)|\, |d\xi_1| \cdots |d\xi_n|}{\prod_{i=1}^n |\xi_i - a_i|^{\alpha_i + 1}}$

$$\le \frac{\alpha!}{(2\pi)^n} \cdot \frac{(2\pi)^n \prod r_i}{r_1^{\alpha_1+1} \cdots r_n^{\alpha_n+1}} \cdot \sup |f(\xi)|$$

Thm (Wierstrass's)

$(f_k) \subset A(\Omega)$. s.t. $f_k \xrightarrow{n.c.c} f \Rightarrow f \in A(\Omega)$.

Besides. $\forall \alpha \in \mathbb{N}^n$. $f_k^{(\alpha)} \xrightarrow{n.c.c} f^{(\alpha)}$.

Pf: 1) By Cauchy formula $\Rightarrow$ expand f.

2) For $a \in \Omega$. $\bar{\Delta}(a, r) \subset \Omega$. $z \in \bar{\Delta}(a, \frac{r}{2})$

$$|f_k^{(\alpha)}(z) - f^{(\alpha)}(z)| \overset{\text{Cauchy}}{\leq} \frac{\alpha!}{(r/2)^\alpha} \cdot \sup_{\bar{\Delta}(z, r/2)} |f_k - f|.$$

$$\Rightarrow \sup_{\bar{\Delta}(a, r/2)} |f_k^{(\alpha)} - f^{(\alpha)}| \leq \frac{\alpha!}{(r/2)^\alpha} \sup_{\bar{\Delta}(a, r)} |f_k - f|$$

So: $f_k^{(\alpha)} \xrightarrow{u} f^{(\alpha)}$ on $\bar{\Delta}(a, r/2)$.

Cover every cpt set by finite nbh.

Thm (Montel)

$(f_k) \subset A(\Omega)$. unif-bdd on every cpt set of Ω. $\Rightarrow \exists (f_{k_n}) \subset (f_k)$. st. $f_{k_n} \xrightarrow{n.c.c} f$.

Pf: For $a \in \Omega$. $\bar{\Delta}(a, r) \subset \Omega$. $M \triangleq \sup_{i, \Omega} |f_i(z)|$.

process as above. by Cauchy:

$$\sup_{k, j, \bar{\Delta}(a, r/2)} \left| \frac{\partial f_k}{\partial z_j} \right| \leq 2M / \min r_j \Rightarrow (f_i) \text{ equi. conti.}$$

Apply Ascoli and diagonal argument.

Thm (Open mapping)

$f \in A(\Omega)$. $f \not\equiv$ const. $\Rightarrow f$ is open map.

Pf: $\forall a \in \Omega$. $B(a,r) \subset \Omega$. $\exists b \in B(a,r)$. s.t.

$f(b) \neq f(a)$. $D =: \{ \zeta \in \mathbb{C}^1 \mid a + \zeta(b-a) \in B(a,r) \}$

$\Rightarrow D$ is open in $\mathbb{C}^1$. $0 \in D$.

Set $h(\zeta) =: f(a + \zeta(b-a)) : D \to \mathbb{C}^1$

So: $h(0) = f(a)$ is interior point in $h(D)$.

Thm. (Maximal modulo principle)

$f \in A(\Omega)$. $f \not\equiv$ const. $\Rightarrow$ $|f(z)|$ can't attain its maximal value in Ω.

Pf: By contradict: $f(\Omega) \subset \{ |w| \leq f(a) \}$.

$\overset{\text{open}}{\underset{\text{map}}{\Rightarrow}}$ $f(\Omega) \subset \{ |w| < f(a) \}$. contradict!

Thm. (Schwarz Lemma)

$f \in A(\Delta^n)$. $f(0) = 0$. $\Rightarrow$ $|f(z)| \leq \max\limits_{\Delta^n} |f| \cdot \|z\|_\infty$ on Δ^n.

Pf: $\varphi(\zeta) = \zeta^{-1} f(a_1\zeta, \cdots a_n\zeta)$. $\zeta \in \Delta$. $a \in \partial\Delta^n$.

By MMP: $|f(a_1\zeta \cdots a_n\zeta)| \leq \max\limits_{\Delta^n} |f|\, |\zeta|$

$\leq \max\limits_{\Delta^n} |f| \, \|(a_1\zeta, \cdots a_n\zeta)\|_\infty$.

(2) <u>Holomorphic in $\mathbb{C}^n \to \mathbb{C}^m$:</u>

For $z = (z_1 \cdots z_n) = (x_1 + iy_1, \cdots x_n + iy_n) \in \Omega_1 \subset \mathbb{C}^n$.

$w = (w_1 \cdots w_m) = (u_1 + iv_1 \cdots u_m + iv_m) \in \Omega_2 \subset \mathbb{C}^m$.

Lemma. For $w = F(z) = (f_1(z), \cdots f_m(z)) : \Omega_1 \to \Omega_2$. g is differentiable for $(x_1, y_1 \cdots x_m, y_m)$.

$$\Rightarrow \frac{\partial (g \circ F)}{\partial z_k}(a) = \sum_{i=1}^{m} \frac{\partial g}{\partial w_i}(F(a)) \frac{\partial f_i}{\partial z_k}(a) + \sum_{i=1}^{m} \frac{\partial g}{\partial \bar{w}_i}(F(a)) \frac{\partial \bar{f_i}}{\partial z_k}(a). \ \forall g \in C^1.$$

Pf: By chain rule in $(u_1, v_1 \cdots u_m, v_m) \subset \mathbb{R}^{2m}$:

$$LHS = \sum_i \frac{\partial g}{\partial u_i}(Fa) \frac{\partial u_i}{\partial z_k}(a) + \sum_i \frac{\partial g}{\partial v_i}(Fa) \frac{\partial v_i}{\partial z_k}(a)$$

replace $u_i = (f_i + \bar{f}_i)/2$. $v_i = (f_i - \bar{f}_i)/2i$

prop. (chain rule)

$F = (f_1 \cdots f_m) : \Omega_1 \to \Omega_2$. $\forall f_i \in A(\Omega_1)$.

If $g \in A(\Omega_2)$. Then. $g \circ F \in A(\Omega_1)$ and.

$$\frac{\partial (g \circ F)}{\partial z_k}(a) = \sum_i^m \frac{\partial g}{\partial w_i}(F(a)) \cdot \frac{\partial f_i}{\partial z_k}(a)$$

Pf: By Lemma above.

Thm. For $F : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$. $F = (f_1 \cdots f_n) \in A(\Omega)$

$F'(z) =: \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{n \times n}$ $\quad J_R f(z) =: \begin{pmatrix} \left(\frac{\partial u_i}{\partial x_j}\right) & \left(\frac{\partial u_i}{\partial y_j}\right) \\ \left(\frac{\partial v_i}{\partial x_j}\right) & \left(\frac{\partial v_i}{\partial y_j}\right) \end{pmatrix}$

$\Rightarrow |F'(z)|^2 = |J_R f(z)|$

Pf: $RHS = \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right|_{\cdot i +} = \left| \begin{pmatrix} \left(\frac{\partial u_i}{\partial x_j} + i \frac{\partial v_i}{\partial x_j}\right) & (\cdots) \\ \left(\frac{\partial v_i}{\partial x_j}\right) & \left(\frac{\partial v_i}{\partial y_j}\right) \end{pmatrix} \right|$

$\overset{\cdot(-i)+}{=} \left| \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \right|$

$\overset{C-R}{=} \left| \begin{pmatrix} \partial F / \partial x & (0) \\ \left(\frac{\partial v_i}{\partial x_j}\right) & \overline{\partial F / \partial x} \end{pmatrix} \right| = |F'(z)|^2$

KOKUYO

Thm (Inverse mapping)

$F: \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$. $F \in A(\Omega)$. If $a \in \Omega$. st.

$|F'(a)| \neq 0$. Then $\exists U_a . U_{F(a)}$ nbd's of

$a. F(a)$. St. $F: U_a \to U_{F(a)}$ one-to-one

and $h =: F^{-1} \in A(U_{F(a)})$. $h'(w) = (F'(z))^{-1}$

for $\forall\ w = F(z)$. $z \in U_a$.

Pf: $|F'(a)| \neq 0 \Rightarrow |J_{\mathbb{R}} F(a)| \neq 0$.

Consider $F: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. $\exists U_a . U_{F(a)}$.

and $h = (g_1 \cdots g_n)$. St. $g_i(F(z)) \equiv z_i$.

$$\underset{\partial/\partial \bar{z}_l}{\overset{\text{Diff}}{\Rightarrow}} \quad \sum_k \frac{\partial g_i}{\partial \bar{w}_k} \cdot \frac{\partial \bar{f}_k}{\partial \bar{z}_l} = 0. \quad \text{for } i.l \leq n.$$

By Cramer's Lemma. $\Rightarrow \dfrac{\partial g_i}{\partial \bar{w}_k} \equiv 0$. on $U_{F(a)}$.

Cor. $F: \Omega \subset \mathbb{C}^n \to \mathbb{C}^m$ $(n \leq m)$. $F \in A(\Omega)$.

If $a \in \Omega$. St. $r(F'(a)) = n$. Then

$\exists U_a \subset \Omega$. $F: U_a \to \mathbb{C}^m$. injective.

Thm. (Implicit Function Thm)

$F = (f_1 \cdots f_m): \Omega \subset \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$. $F \in A(\Omega)$.

If $(z_0, w_0) \in \Omega$. St. $F(z_0, w_0) = 0$. $\left(\dfrac{\partial f_k}{\partial w_j}(z_0, w_0)\right)_{m \times m}$

has full rank. Then $\exists U = U_1 \times U_2$ nbd of

(z_0, w_0). and $g: U_1 \to U_2$. St. $g \in A(U_1)$ and

$\{(z, w) \in U \mid F(z, w) = 0\} = \{(z, g(z)) \mid z \in U_1\}$

Pf: Set $G(z,w) = (z, F(z,w)) : \Omega \to \mathbb{C}^n \times \mathbb{C}^m$.

$\Rightarrow |G'(z_0, w_0)| \neq 0$. $\exists\, G^{-1}(u,v) = (u, h(u,v))$

Set $g(z) =: h(z,0)$ is what we need.

Thm. (Maximal Module Principle)

$f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^m$. $f \in A(\Omega)$. If $|f(a)|$ is max of $|f(z)|$. $a \in \Omega$ Then: $|f| \equiv \text{const.}$ in Ω.

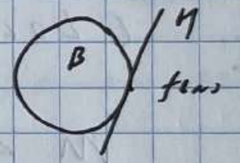
Pf. $B =: \{ w \in \mathbb{C}^m \mid |w| \le |f(a)| \}$ convex. $f(a) \in \partial B$.

If $f(\Omega) \subset B$. Then. $\exists$ plane:

H: $\mathrm{Re}\, L(w) = \mathrm{Re}\, L(f(a))$, $L(w) = \sum_1^m c_k w_k$.

St. $H \cap B \subseteq \partial B$.

$\mathrm{Re}\, L(w) \le \mathrm{Re}\, L(f(a))$. $w \in B$.



$$\Rightarrow |e^{Lf(z)}| = e^{\mathrm{Re}(Lf(z))} \le e^{\mathrm{Re}(Lf(a))}$$

$\therefore$ $Lf(z) \equiv \text{const.}$ in Ω. $\Rightarrow \mathrm{Re}\, Lf(z) \equiv \mathrm{Re}\, Lf(a)$.

So: $f(\Omega) \subset H \cap B \subset \partial B$. $|f(z)| \equiv |f(a)|$.

Rmk: Note B is strictly convex $\Rightarrow H \cap B = \{f(a)\}$.

So: $f(z) \equiv f(a)$ in Ω.

But if we use $|\cdot|_\infty$. $\# H \cap B$ may > 1.

(3) Submanifolds in $\mathbb{C}^n$:

Def. $Z \subset \mathbb{C}^n$ is complex submanifold of $\dim = n-q$.

if $\forall\, w \in Z$. $\exists\, w \overset{\text{nbd}}{\subset} W \subset \mathbb{C}^n$. $f_i \in A(W)$. s.t.

$Z \cap W = \{ z \in W \mid f_i(z) = 0 .\ 1 \le i \le q \}$. $\mathrm{rk}\left(\left(\frac{\partial f_i}{\partial z_j} \right)_{n \times q} \right) = q$.

Rmk: It means: by IFT. $Z \cap W = \{ (z_1 \cdots z_{n-q},$

$h_1(z_1 \cdots z_{n-q}) \cdots h_q(z_1 \cdots z_{n-q}) \}$

Lemma. $(n-q)$-dim complex submanifolds in $\mathbb{C}^n \Leftrightarrow$

$(2n-2q)$-dim smooth real submanifolds in $\mathbb{R}^{2n}$

Pf: Note $f_1 = \cdots = f_q = 0 \Leftrightarrow$

$u_1 = v_1 = \cdots u_q = v_q = 0$

$$\left| \left(\frac{\partial f_i}{\partial z_j} \right)_{q \times q} \right|^2 = \left| \begin{pmatrix} \left(\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}\right) \\ \left(\frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}\right) \end{pmatrix} \right| \neq 0$$

Thm. (Local parametrization)

$M \subset \mathbb{C}^n$ is p-dim complex submanifold

If $z \in M$. Then $\exists$ nbd U_z of z. $r \subset U_r$

$\subset \mathbb{C}^n$. $F: U_r \xrightarrow[\text{Holo.}]{\sim} U_z$. s.t. $M \cap U_z =$

$F\{ (z_1 \cdots z_n) \in U_r \mid z_1 = \cdots z_{n-p} = 0 \}$

Pf: By IFT. $F = h(z, 0)$ as def in pf of IFT.

set $H =: (h_1(z, w) \cdots h_{n-p}(z, w))$

Def: For connected complex submanifold in $\mathbb{C}^n$ with

dim $= k$. Denote it by M. Define:

$f: M \to \mathbb{C}^1$ is holomorphic if $\forall p \in M$. $\exists U_p$

nbd of p. $F \in A(U_p)$. s.t. $F|_{M \cap U_p} = f|_{M \cap U_p}$.

Thm. If $f: M \to \mathbb{C}^1$ is holomorphic. $\exists p \in M$. nbd U_p of p. st. $f|_{M \cap U_p} = 0$. Then $f \equiv 0$ on M.

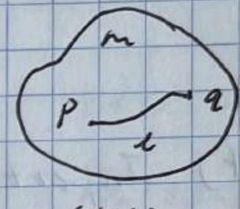
Pf: For point p. $\exists \widetilde{U}_p$ and F. st.

$F|_{M \cap U_p \cap \widetilde{U}_p} = f|_{M \cap U_p \cap \widetilde{U}_p} = 0$.

By parametrization Thm. $\exists h$. st. ....

st. $F \circ h^{-1}(0, \dots, 0, w) = 0$ on open set.

$\Rightarrow F|_M \equiv 0$.



Next. for $\forall q \in M$. $\exists l$. st.

Cover l by finite such nbd's. (U_i)

st. $U_i \cap U_{i+1} \neq \emptyset$. $\Rightarrow$ corresp. (F_i): $F_i|_{U_i \cap U_{i+1}} = F_{i+1}|_{\square} = 0$

$\Rightarrow f(p) = f(q) = 0$. i.e. $f \equiv 0$ on M.

Thm. If $f: M \to \mathbb{C}^1$. holomorphic. $f \not\equiv$ const.

Then $|f|$ can't attain max on M.

Pf: If $|f|$ attain max at p.

Argue as above. $\exists U_p$. $F \in A(U_p)$.

$\Rightarrow |F|$ also attain max at p $\Rightarrow$ $F|_{U_p} \equiv$ const

Find connected path from p to $q \in M$.

The corresp. $(F_i)_1^k$ on this path are const.

Cor. For M is cpt connected submanifold in $\mathbb{C}^n$. $f: M \to \mathbb{C}^1$. holomorphic

$\Rightarrow f \equiv$ const. on M.

Cor. $\forall$ connected cpt complex submanifold in $\mathbb{C}^n$ is a point.

Pf: $z_i : (z_1 \cdots z_n) \in M \mapsto z_i \in \mathbb{C}^1$ is holomorphic.

If $\exists\, p, q \in M$. s.t. $p \neq q$.

Then. $\exists\, k$. s.t. $p_k \neq q_k$.

$\Rightarrow z_k(p) \neq z_k(q)$. contradict!

(4) Injective holomorphic:

Thm. $D \subset \mathbb{C}^n$. $f \in A(D)$. $f \not\equiv 0$. For $U \subset_{open} D$.

Set $Z(f, U) =: \{z \in U \mid f(z) = 0\}$

If $Z(f, D) \neq \emptyset$. Then $\exists\, V \subset D$. st. $Z(f, V)$ is $(n-1)$-dim complex manifold in V.

Cor. $\forall\, p \in Z(f, D)$. with nbd U_p. $\exists\, q \in Z(f, U)$. nbd $V_q \subset U$. $g \in A(V_q)$. st.

i) $Z(f, V_q) = Z(g, V_q)$

ii) $dg(z) \neq 0$ on V. i.e. $r((\frac{dg}{dz_i})) = 1$.

Rmk: There's no need that $df(p) \neq 0$ in D.

e.g. $f(z) = z_1^2$.

Pf: $\lambda =: \max\{\lambda \geq 0 \mid (\frac{\partial}{\partial z})^\alpha f(z) = 0$ on $Z(f, D)$.

$\alpha \in \mathbb{N}^n$. $|\alpha| \leq \lambda\}$. $< \infty$. since $f \not\equiv 0$.

$\Rightarrow \exists \beta \in \mathbb{N}^n$. s.t. $|\beta| = \Delta$. $p_0 \in Z(f, D)$. s.t.

$\lambda((\frac{\partial}{\partial z})^\beta f)(p) \neq 0$. on V_{p_0} nbd of p_0.

$\Rightarrow Z((\frac{\partial}{\partial z})^\beta f, V_{p_0})$ is $(n-1)$-dim manifold.

By local para. WLOG. $Z((\frac{\partial}{\partial z})^\beta f, W) = \{w \in W | w_n = 0\}$. where $W \subset V_{p_0}$. and $p_0 = 0$.

Note: $Z(f, W) \subset Z((\frac{\partial}{\partial z})^\beta f, W)$.

prove: $Z(f, W) = Z((\frac{\partial}{\partial z})^\beta f, W)$.

Find δ_n. s.t. $f(0, w_n)$ has unique zero at $w_n = 0$ (with multinumber k) on $\{|w_n| \leq \delta_n\}$.

Note: $|f(\tilde{w}, w_n) - f(0, w_n)| \xrightarrow{\tilde{w} \to 0} 0$

$\exists \tilde{\delta}$. s.t. $|f(\tilde{w}, w_n) - f(0, w_n)| < \inf_{|w_n| = \delta_n} |f(0, w_n)|$

when $\tilde{w} \in P(0, \tilde{\delta}) \subseteq \mathbb{C}^{n-1}$.

By Rouche Thm:

$\forall \tilde{w} \in P(0, \tilde{\delta})$. (fixed). $f(\tilde{w}, w_n)$ has k zeros in $\{|w_n| \leq \delta_n\}$. i.e. $\exists w_n'$. $\in \{\cdots\}$. s.t. $(\tilde{w}, w_n') \in Z(f, W) \subset Z(\cdots)$

$\Rightarrow w_n' = 0$. i.e. $Z(f, W) = Z((\frac{\partial}{\partial z})^\beta f, W)$.

Thm. $F = (f_1 \cdots f_n): D \subset \mathbb{C}^n \to \mathbb{C}^n$. is injective holomorphic $\Rightarrow \det(\frac{\partial f_i}{\partial z_j})_{n \times n}(z) \neq 0$ on D.

Cor. $F \in A(D)$ is locally injective in nbd of $z_0 \Leftrightarrow \det(\frac{\partial f_i}{\partial z_j})_{n \times n}(z_0) \neq 0$.

Pf: Induct on n. $n=1$ we have proved.

Set $h(z) = \det\left(\frac{\partial f_i}{\partial z_j}(z)\right)_{n\times n} \in A(D)$.

By contradict: if $Z(h, D) \neq \emptyset$.

$\Rightarrow \exists M \subset Z(h, D)$ is $(n-1)$-dim manifold.

Next. prove: $\left(\frac{\partial f_i}{\partial z_j}(z)\right)_{n\times n} \equiv 0$ on M.

Otherwise. $\exists a \in M$. St. $\left(\frac{\partial f_i}{\partial z_j}(a)\right) \neq 0$.

WLOG. $\frac{\partial f_n}{\partial z_n}(a) \neq 0$.

Set $W(z) = (z_1 \cdots z_{n-1}, f_n(z))$.

$\Rightarrow \frac{\partial w}{\partial z} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & \partial f_n/\partial z_n \end{pmatrix} \neq 0$ at $z = a$.

So W^{-1} holomorphic exist. in nbd of $W(a)$

Set $\widetilde{F}(w) = F \circ W^{-1} = (g_1(w) \cdots g_{n-1}(w), w_n)$

$G(\widetilde{w}) = (g_1(\widetilde{w}, W(a_n)) \cdots g_{n-1}(\widetilde{w}, W(a_n)))$

$\Rightarrow G$ is holomorphic and injective in nbd of $\widetilde{W}(a)$.

By inductive hypo: $\left|\left(\frac{\partial g_i}{\partial w_j}(\widetilde{W}(a), W(a_n))\right)\right| \neq 0$

So: $\left|\left(\frac{\partial f_i}{\partial z_j}(a)\right)\right| = \left|\left(\frac{\partial g_i}{\partial w_j}(\widetilde{W}(a), W(a_n))\right)\right| \cdot$

$\left|\left(\frac{\partial w_i}{\partial z_j}(a)\right)\right| \neq 0$

contradict with $a \in Z(h, D)$

So: $F \equiv$ const on $M \neq \emptyset$. contradict with F is injective.

(5) Biholomorphism on Domains:

① For $\mathbb{C}^n$:

Recall when $n=1$. $\mathrm{Aut}(\mathbb{C}^1) = \{az+b \mid a \neq 0, b \in \mathbb{C}\}$

But when $n \geq 2$. There's a huge group of Aut of $\mathbb{C}^n$. It follows from a phenomenon:

When $n \geq 2$. $\exists$ biholomorphism from $\mathbb{C}^n$ to its proper domain. We call such map / domain by:

Fatou - Bieberbach mapping / domain.

e.g. set $F(z_1, z_2) = \frac{1}{2}(z_2, z_1 + z_2^2)$. which is invercible and has unique fix point 0.

set the basin of attraction of origin $\tilde{B} =: \bigcup_{k \geq 0} (F^{-1})^{(k)}(B^2)$. Then:

$\tilde{B}$ is a Fatou - Bieberbach domain.

Thm. (general)

$\forall F \in \mathrm{Aut}(\mathbb{C}^n)$. with an attracting fix point at origin (i.e. Absolute value of every eigenvalue of the Jacobian matrix of $F < 1$).

$\Rightarrow$ Its basin of attraction of fix point is biholo- equi. to $\mathbb{C}^n$.

Rmk: If it's a proper set of $\mathbb{C}^n$.

Then it's Fatou - Bieberbach.

② Bdd domains in $\mathbb{C}^n$:

i) Lemma. (Cartan)

$f: D \to D$. $f \in A(D)$. If $\exists a \in D$. s.t. $f(a) = a$. $f'(a) = I_n$. Then $f \equiv z$.

Pf: Assume $a = 0$. $D \subset \Delta(0, R)$.

$f(z) = z + P_N(z) + O(|z|^{N+1})$.

$P_N(z)$ is homogenous poly of degree $= N$.

By Cauchy estimate, if $f(z) =$

$(\sum a_{1,\alpha} z^\alpha, \cdots, \sum a_{n,\alpha} z^\alpha)$.

$\Rightarrow |a_{k,\alpha}| \le R / \varepsilon^{\alpha}$. on $\bar{\Delta}(0, \varepsilon) \subset D$.

Note $f^{(k)}(z) = z + k P_N(z) + O(|z|^{N+1})$

$f^{(k)}(D) \subset D$. $\forall k$.

Suppose $P_N(z) = (\sum_{|\alpha|=N} b_{1,\alpha} z^\alpha, \cdots \sum_{|\alpha|=N} b_{n,\alpha} z^\alpha)$

$\Rightarrow \forall k$. $|k \cdot b_{j,\alpha}| \le R/\varepsilon^\alpha$. $\therefore P_N(z) \equiv 0$

Thm. (Cartan)

$0 \in D \subset \mathbb{C}^n$. $\forall z \in D$. $\Rightarrow e^{i\theta} z \in D$. $\forall \theta$.

If D_1, D_2 are such domain. $f(0) = 0$. where $f: D_1 \xrightarrow{\sim} D_2$. biholomorpic. Then f is linear.

Pf: Set $\varphi = f^{-1}$. $g = e^{-i\theta}\varphi(e^{i\theta}f(z))$

$\Rightarrow g(0) = 0$. $g'(0) = I_n$.

By Cartan Lemma. $f(e^{i\theta}z) = e^{i\theta}f(z)$

Expand in series and compare the coefficients : $a_\alpha e^{i\theta} = a_\alpha e^{i\theta|\alpha|}$

$\Rightarrow$ i.e. $|\alpha| \neq 1 \Rightarrow a_\alpha = 0$.

Lemma. If $\varphi \in Aut(B^n)$. $\varphi(0) = 0$. Then exists. unitary matrix T. s.t. $\varphi(z) = zT$.

Pf: By Cartan Thm $\Rightarrow \varphi$ is linear.

Set $\varphi(z) = zT$.

For $z_0 \in B^n$. if $|z_0| < |z_0 T|$.

then : $|z_0|/|z_0T| \in B^n$. $\varphi(|z_0|/|z_0T|)$

$= \frac{z_0 T}{|z_0 T|} \in B^n$. contradict!

if $|z_0| > |z_0T|$. argue similarly.

Def: $\Omega \subset \mathbb{C}^n$ is homogeneous domain if $\forall p, q \in \Omega$. $\exists \varphi \in Aut(\Omega)$. s.t. $\varphi(p) = q$.

Remark: $\varphi_\beta : B^n \to B^n$. $z \mapsto \left(\frac{z_1 - \beta}{1-\bar\beta z_1}, \frac{\sqrt{1-|\beta|^2}\, z_2}{1-\bar\beta z_1}, \cdots \frac{\sqrt{1-|\beta|^2}\, z_n}{1-\bar\beta z_1}\right)$

Rmk: $\varphi_\beta^{-1}(u) = \left(\frac{u_1+\beta}{1+\bar\beta u_1}, \frac{u_2\sqrt{1-|\beta|^2}}{1+\bar\beta u_1}, \cdots \frac{u_n\sqrt{1-|\beta|^2}}{1+\bar\beta u_1}\right)$

$\varphi_\beta, \varphi_\beta^{-1} \in A(B^n)$. Besides. $\varphi_\beta(\beta, 0 \cdots 0) = 0$.

Thm. B^n is homogeneous domain.

$\forall \phi \in Aut(B^n)$, can be decomposed in

$\phi = U_1 \circ \varphi_{\phi(0)}^{-1} \circ U_2$, where U_1, U_2 are unitary matrixs.

Pf: $\forall w \in B^n$. Chose $U = \left(\begin{matrix} \frac{w_1}{|w|} \\ \vdots \\ \frac{w_n}{|w|} \end{matrix} \; \beta_2^T \cdots \beta_n^T \right)$

unitary. $\Rightarrow \varphi_\beta \circ U(w) = 0$.

Since $\beta_i \perp w$, $\forall i \geq 2$.

Note $\varphi_\beta \circ U \in Aut(B^n)$.

So: we can chose $\varphi_1, \varphi_2 \in Aut(B^n)$.

s.t. $\varphi_1(\zeta_1) = 0$, $\varphi_2(\zeta_2) = 0$.

For decompose:

Chose $U_1^{-1} \circ \phi(0) = (|\phi(0)|, 0 \cdots 0)$.

Then apply the Lemma above.

ii) **Thm.** Δ^n is homogeneous domain.

$Aut(\Delta^n) = \{ f(z) = (e^{i\theta_1} \frac{z_{\sigma(1)} - \alpha_1}{1 - \bar{\alpha}_1 z_{\sigma(1)}}, \cdots, e^{i\theta_n} \frac{z_{\sigma(n)} - \alpha_n}{1 - \bar{\alpha}_n z_{\sigma(n)}}) \mid z \in S_n, \theta_i \in \mathbb{R}^1, \alpha \in \Delta^n \}$

Pf: Set $\sigma_\alpha(z) = \left(\frac{z_1 - \alpha_1}{1 - \bar{\alpha}_1 z_1} \cdots \frac{z_n - \alpha_n}{1 - \bar{\alpha}_n z_n} \right)$

So $\sigma_\alpha \in Aut(\Delta^n)$, $\sigma_\alpha(\alpha) = 0$.

Set $f = \sigma_{f(0)} \circ f$. $\Rightarrow f(0) = 0$.

By Cartan: $f(z) = (\sum_{k=1}^{n} a_{1k} z_k, \dots \sum_{k=1}^{n} a_{nk} z_k)$

Note $f(\Delta^n) \subset \Delta^n \Rightarrow |\sum_k a_{ik} z_k| < 1$.

choose $z_k = \frac{\bar{a}_{ik}}{|a_{ik}|} r$. $0 < r < 1$. $\overset{r \to 1^-}{\Rightarrow} \sum_k^n |a_{ik}| \le 1$.

choose $p_\nu = (0, 0 \dots 1 - \frac{1}{\nu}, 0 \dots) \overset{\nu \to 0}{\to} \partial \Delta^n$.

So $f(p_\nu) \to \partial \Delta^n$. i.e. $\max_{1 \le i \le n} |a_{ik}| = 1$.

$\Rightarrow f(z) = (e^{i\theta_1} z_{\sigma(1)}, \dots e^{i\theta_n} z_{\sigma(n)})$.

iii) <u>Lemma.</u> $f_k \in A(\Omega)$. $k \le m$. If $\sum |f_k|^2 \equiv$ const. on Ω. Then $f_k \equiv \text{const}_k$. $\forall k$

<u>Pf:</u> $\frac{\partial^2}{\partial z_k \partial \bar{z}_k} \sum |f_i|^2 = \sum |\frac{\partial f_i}{\partial z_k}|^2 = 0$

<u>Lemma.</u> $g: B^m \to \Delta^m$. $f: \Delta^m \to B^m$. holomorphic. s.t. $g(0) = 0$. $f(0) = 0$. Then:

$|g(z)|_\infty \le |z|$. $|f(z)| \le |z|_\infty$.

<u>Pf.</u> 1) f is from Schwartz Lemma.

2) Apply MMP on $g_k(z)/|z|$. each k.

<u>Thm.</u> (Poincaré inequi.)

$n \ge 2$. There's no biholo: $\Delta^n \underset{\varphi}{\xrightarrow{\sim}} B^n$.

<u>Pf:</u> By Lemma. $\Rightarrow |\varphi| = |z|_\infty$.

$\Rightarrow \sum |\frac{\varphi_k(z)}{|z|_\infty}| = 1$.

So: $\varphi_k(z) = c_k \cdot |z|_\infty$ follows from the 1^{st} Lemma.

$\Rightarrow \varphi(z) = |z|^n(c_1 \cdots c_k)$. which's not bijection. Contradiction!

Rmk: i) The inequivalence is from the strong difference of their boundaries: $\partial\Delta^n$ can contain a real line But ∂B^n can't!

ii) Δ^n and B^n are diffeomorphism when they are viewed as domains in $\mathbb{R}^{2n}$.

Thm. If a bdd simply connected domain in $\mathbb{C}^n$ having a connected. smooth boundary that's spherical (i.e. locally biholo- to piece of a ball)

Then it's biholo-equi. to a ball.