Bergman Metric.

(1) Estimate:

<u>Def</u>: i) For $z = (z_1 \cdots z_n) = (x_1 + i x_2 \cdots x_{2n-1} + i x_{2n})$

denote $dV = dx_1 \wedge dx_2 \cdots dx_{2n} = (\frac{i}{2})^n \cdot$

$dz_1 \wedge d\bar{z}_1 \cdots \wedge d\bar{z}_n \wedge dz_n$.

ii) $L^2(\Omega) =: \{ f : \Omega \to \mathbb{C} \mid f$ is $\mathbb{C}$-measurable

and $\int_\Omega |f(z)|^2 dz < \infty \}$

equipped with $\langle f, g \rangle =: \int_\Omega f \bar{g} \, dV$.

<u>Rmk</u>: $L^2(\mathbb{C}^n) = \{0\}$. $n \geq 1$.

<u>Pf</u>: By proj. only prove $n = 1$.

$$\int_{\mathbb{C}} |f|^2 = \int_0^\infty \int_0^{2\pi} |f(re^{i\theta})|^2 r \, d\theta \, dr$$

$$= \sum \int_0^\infty \int_0^{2\pi} \sum r^{k+j} e^{i\theta - j\theta} r \, d\theta \, dr$$

$$\overset{\text{ortho.}}{=} \sum_{k \geq 1} \int_0^\infty 2\pi r^{2k} dr = \infty$$

<u>Rmk</u>: Similarly, we can prove: it holds when Ω can contain a sector.

iii) $A^2(\Omega) =: L^2(\Omega) \cap A(\Omega)$.

Next, we want to prove $A^2(\Omega)$ is CLS of $L^2(\Omega)$.

Thm. (Mean Value property)

$\overline{B(a,R)} \subset \Omega \subset \mathbb{C}^n$. $f \in A(\Omega)$. Then:

$$f(a) = ⨍_{\partial B(a,R)} f(z)\,dz = ⨍_{B(a,R)} f\,dz.$$

Pf: $f(a) = \frac{1}{|\partial B(0,R)|}\int_{\partial B(0,R)} \left(\frac{1}{2\pi}\int_0^{2\pi} f(a+e^{i\theta}\zeta)\,d\theta\right) ds(\zeta)$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \left(\frac{1}{|\partial B_R|}\int_{\partial B_R} f(a+e^{i\theta}\zeta)\,ds(\zeta)\right)$$

$$= ⨍_{\partial B(0,R)} f(a+\zeta)\,ds(\zeta).$$

Cor. (Cauchy estimate)

$f \in A(\Omega)$. If $\overline{B(a,R)} \subset \Omega$. Then.

$$|f(a)|^2 \le ⨍_{B(a,R)} |f|^2\,dV.$$

Pf: By Hölder inequi.

Thm. $\Omega \subset \mathbb{C}^n$. $A^2(\Omega)$ is CLS of $L^2(\Omega)$.

Pf: If (f_n) is Cauchy in $L^2(\Omega)$.

Then $\exists f_0 \in L^2(\Omega)$. s.t. $\|f_n - f_0\|_2 \to 0$.

For $a \in \Omega$. $\overline{B(a,R)} \subset \Omega$.

$\forall z \in B(a,\frac{R}{2})$. by Cauchy estimate:

$$|f_i(z) - f_j(z)|^2 \le ⨍_{B(z,R/2)} |f_i - f_j|^2\,d\zeta.$$

$$\lesssim \|f_i - f_j\|_{L^2}^2$$

$\Rightarrow (f_i)$ is Cauchy (unif.) in $B(a,\frac{R}{2})$.

So $\exists\, g \in A(\Omega)$. $f_n \xrightarrow{u.c.c} g$. $g = f$.

<u>Cor.</u> $A^2(\Omega)$ is Hilbert space.

(2) <u>Bergman Kernel:</u>

<u>Lemma.</u> For $a \in D$. $Z_a : \begin{matrix} A^2(D) \to \mathbb{C} \\ f \mapsto f(a) \end{matrix}$ is BLO.

<u>Pf:</u> $|Z_a(f)| \overset{\text{Cau.}}{\le} |B_R|^{-\frac{1}{2}} \|f\|_{L^2(\Omega)}$.

where $R = d(a, \partial D)$.

<u>Cor.</u> $\forall\, K \subset D$. cpt. $\sup\limits_{K} \|Z_a\| < \infty$.

<u>Pf:</u> By UBP.

<u>Def:</u> By Riesz repre. $\exists!\, K_D(\cdot, a) \in A^2(D)$. s.t.

$Z_a(f) = \langle f, K_D(\cdot, a) \rangle = \int_D \overline{K(z,a)}\, f(z)\, dz$.

$\|K_D(\cdot, a)\| = \|Z_a\| = \sup\{|f(a)| \mid \|f\| \le 1.\ f \in A^2(D)\}$.

$K_D : D \times D \to \mathbb{C}$ is called Bergman kernel of D.

<u>Rmk:</u> $\|K_D(\cdot, a)\|_{L^2(D)} \lesssim (d(a, \partial D))^{-n}$. $a \in D$.

<u>Lemma.</u> $K_D(\zeta, z) = \overline{K_D(z, \zeta)}$

<u>Pf:</u> LHS $= \langle K_D(\cdot, z), K_D(\cdot, \zeta) \rangle$

$= \overline{\langle K_D(\cdot, \zeta), K_D(\cdot, z) \rangle} =$ RHS.

Lemma. $D \subset \mathbb{C}^n$. bdd domain.

i) $\forall z \in D$. $K_D(z,z) = \sup\{|f(z)|^2 : f \in A^2(D).\ \|f\|=1\}$

ii) $K_D(z,z) \geq 1/|D|$.

Pf: i) $K_D(z,z) = \|K_D(\cdot, z)\|^2 = \|\tau_z\|^2$.

ii) Choose $f(z) = 1/|D|^{\frac{1}{2}}$

Lemma. $D_1 \subset D_2 \subset \mathbb{C}^n$. bdd domain. Then.

$K_{D_2}(z,z) \leq K_{D_1}(z,z)$. $\forall z \in D_1$

Pf: Note $f \in A^2(D_2)$. $\|f\|_{L^2(D_2)} = 1$.

$\Rightarrow f I_{\{z \in D_1\}} \in A^2(D_1)$. $\|f\|_{L^2(D_1)} \leq 1$

Rmk: Note $K_D(z,z) = \|\tau_z^D\|^2$. We have some estimate of norm of evaluation map τ_z^D.

Lemma. $K \subset D$. cpt. Then $\exists\, C_K < \infty$. s.t. for any o.n.b. $(e_i) \subset A^2(D)$. We have:

$$\sup_{z \in K} \sum_{j \geq 1} |e_j(z)|^2 \leq C_K.$$

Pf: Note $\|K_D(\cdot, z)\|^2_{L^2(D)} \leq C_n (d(K, \partial D))^{-n} =: C_K$

But $LHS = \sum |\langle K_D(\cdot, z), e_j\rangle|^2$

$= \sum |e_j(z)|^2$.

prop. (Represent of Bergman kernel)

$(\varphi_i) \subset A^2(D)$ is o.n.b. Then:

$K_D(\zeta, z) = \sum_{j=1} \varphi_j(\zeta)\overline{\varphi_j}(z)$. on $D\times D$. uni. conv g.

on every cpt set of $D\times D$. it's also real

analytic function (can be expanded as series

of z and $\bar{z}$)

Pf: 1°) $K_D(\zeta, z) = \sum (K_D(\cdot, z), \varphi_j)\varphi_j(\zeta)$

$= \sum \overline{\varphi_j}(z)\varphi_j(\zeta)$.

2°) By Lemma above it uniformly bdd

on every cpt set.

(RHS $\le (\sum|\varphi_i|^2)^{\frac{1}{2}}(\sum|\varphi_i|^2)^{\frac{1}{2}} \le C(K)$

Besides. $\sum_1^n \varphi_i(\zeta)\overline{\varphi_i(\bar{z})}$ is holo-.

Apply Weierstrass and Montel Thm

$\Rightarrow K_D(\zeta, \bar{z})$ is holomorphic.

Cor. $K_D \in C^\infty(D\times D)$.

Lemma $P_D: L^2(D) \to A^2(D)$ is canonical proj.

Then: $P_D f(z) = (f, K_D(\cdot, z)) = \int_D f(\zeta)K_D(\zeta, z)\,dV(\zeta)$.

Pf: $\forall f \in L^2(D)$: LHS $= (P_D f, P_D K_D(\cdot, z))$

$= (f, P_D K_D(\cdot, z)) =$ RHS.

<u>Lemma</u> $D_1 \subset \mathbb{C}^{n_1}$. $D_2 \subset \mathbb{C}^{n_2}$. bdd domains.

with Bergman kernels K_{D_1}. K_{D_2}.

Then the Bergman kernel of $D_1 \times D_2$

is $K_{D_1}(\zeta_1, z_1) K_{D_2}(\zeta_2, z_2)$.

<u>Pf</u>: $K_{D_1}(z_1, \lambda_1) K_{D_2}(z_2, \lambda_2) =$

$\int_{D_1 \times D_2} K_{D_1}(\zeta_1, \lambda_1) K_{D_2}(\zeta_2, \lambda_2) K_{D_1 \times D_2}((z_1, z_2), (\zeta_1, \zeta_2))$

$= \int_{D_1} K_{D_1} (\int_{D_2} \cdots) = \cdots = K_{D_1 \times D_2}(z_1, z_2, \lambda_1, \lambda_2)$

<u>Cor.</u> $K_{\Delta^n}(z, w) = \frac{1}{\pi^n} \prod_1^n 1/(1 - z_j \bar{w}_j)^2$.

<u>Pf</u>: Check $K_\Delta(z, w) = \frac{1}{\pi} \cdot 1/(1 - z\bar{w})^2$.

Note $(\sqrt{\frac{k+1}{\pi}} z^k)$ is o.n.b of $A^2(\Delta)$.

<u>Lemma.</u> $F: D_1 \xrightarrow{\sim} D_2$. biholo between bdd domains.

$\Rightarrow K_{D_1}(\zeta, z) = \det F'(\zeta) K_{D_2}(F(\zeta), F(z)) \overline{\det F'(z)}$

<u>Pf</u>: Set $T_F: \begin{array}{ccc} L^2(D_2) & \longrightarrow & L^2(D_1) \\ f & \longmapsto & (f \circ F) \cdot \det F' \end{array}$

is isometry. Since:

$\int_{D_2} |f|^2 = \int_{D_1} |f \circ F(\zeta)|^2 \cdot |\det F'(\zeta)|^2 dV(\zeta)$.

Besides. $T_F^{-1} = T_{F^{-1}}$.

$(T_{F^{-1}} f, K_{D_2}(\cdot, w))_{D_2} \overset{iso}{=} (f, T_F \circ K_{D_2}(\cdot, w))_{D_1}$

$= T_{F^{-1}} f(w) = f(z) \cdot |F'(z)|^{-1}$.

<u>Cor.</u> $\forall f \in L^2(D_2)$. $T_F(P_{D_2} f) = P_{D_1}(T_F f)$.

<u>Pf:</u> RHS $= \langle T_F f, K_{D_1}(\cdot, z)\rangle_{D_1}$.

replace $K_{D_1}(z, \cdot)$ by $\cdots$ above.

<u>Rmk:</u> We can obtain $K_{B^n}(z,\zeta) = \frac{n!}{\pi^n} \cdot 1/(1-z\bar{\zeta})^{n+1}$

from the Lemma.

Note: $K_{B^n}(\cdot, 0) = 1/|B^n|$. by MVT.

Set $F(w) = U \circ \varphi^{-1}_{|\zeta|}(w)$. where U

is unitary. St. $U(|\zeta|, 0 \cdots 0) = \zeta$.

(3) <u>Bergman Metric:</u>

Zz's generalization of Pioncare metric on $\mathbb{C}^1$.

<u>Thm.</u> For $\Omega \underset{bdd}{\subseteq} \mathbb{C}^n$. $\forall (\varphi_k)$ o.n.b. of $A^2(\Omega)$.

$\Rightarrow K_\Omega(z,z) = \sum_k |\varphi_k|^2 > 0$. Besides,

$\left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K_\Omega(z,z)\right)_{n\times n}$ is smooth and

positive definite.

<u>Pf:</u> 1) If $K_\Omega(z_1, z_1) = 0 \Rightarrow \forall k$. $\varphi_k(z_1) = 0$.

So: $K_\Omega(\zeta, z_0) = 0$. $\forall \zeta$.

$\Rightarrow f(z_1) = 0$. $\forall f \in A^2(\Omega)$. contradict!

2) Set $\eta_i(z) =: \varphi_i'(z) \cdot \zeta$.

check: $K_\Omega^2(z,z)\sum_{i,j}^{n}\frac{\partial^2\log K_\Omega(z,z)}{\partial z_i\partial\bar z_j}\xi_i\bar\xi_j$

$= \sum|\varphi_j(z)|^2\sum|\eta_l(z)|^2 - |\sum \eta_l\varphi_l(z)|^2$

$\overset{|\xi|\neq 0}{\geq} 0$. "=" holds when $\varphi_i/\eta_i = k_0\ \forall i$

But K_Ω is indept of choice of (φ_k).

Set $\varphi_1 = c$. $\varphi_2'(z_0)\cdot\xi = \eta_2(z_0)\neq 0$.

So they're l.i. $\Rightarrow$ LHS >0.

Def: i) Bergman metric w.r.t Ω. is def by: ds^2

$=: \sum_{i,j}^{n}\frac{\partial^2\log K_\Omega(z,z)}{\partial z_i\partial\bar z_j}\cdot dz_i\,d\bar z_j$.

Rmk: For $\gamma:[0,1]\to\Omega$. C^1-curve. Then,

$|\gamma|_{Berg} = \int_0^1\left(\sum_{i,j}^{n}\frac{\partial^2\log K_\Omega(\gamma(t),\gamma(t))}{\partial z_i\partial\bar z_j}\gamma_i'(t)\bar\gamma_j'(t)\right)^{\frac12}dt$

ii) For $z_1, z_2\in\Omega$. Then we define Bergman

dist. $\delta_{Berg}(z_1,z_2) =: \inf\{|\gamma|_{Berg} \mid \gamma:[0,1]\to\Omega$.

is C^1. $\gamma(0)=z_1$. $\gamma(1)=z_2\}$.

Thm, $\Omega_1, \Omega_2 \underset{bdd}{\subseteq}\mathbb{C}^n$. $f:\Omega_1\xrightarrow{\sim}\Omega_2$. biholo.

Then. $\sum_{k,l}^{n}\frac{\partial^2\log K_{\Omega_2}(f(z),f(z))}{\partial w_k\partial\bar w_l}\sum_i\frac{\partial f_k(z)}{\partial z_i}\xi_i\sum_j\overline{\frac{\partial f_l(z)}{\partial z_j}}\bar\xi_j$

$= \sum_{i,j}^{n}\frac{\partial^2\log K_{\Omega_1}(z,z)}{\partial z_i\partial\bar z_j}\xi_i\bar\xi_j$. $w=f(z)$.

Pf: Note $K_{\Omega_1}(z,z) = K_{\Omega_2}(f(z), f(z)) \, |f'(z)| \cdot |\bar{f'(z)}|$

and $\dfrac{\partial^2 \log f\bar{f}}{\partial z \partial \bar{z}} = 0. \quad \forall f \in A(\Omega)$.

Apply chain rule on RHS.

Cor. $\Omega_1, \Omega_2 \subset_{bdd} \mathbb{C}^n$. $f: \Omega_1 \xrightarrow{\sim} \Omega_2$. biholo.

i) If $\gamma: [0,1] \to \Omega_1$. C^1-curve. Then:

$|\gamma|_{B(\Omega_1)} = |f(\gamma)|_{B(\Omega_2)}$.

ii) $z_1, z_2 \in \Omega_1$. $\Rightarrow \delta_{B(\Omega_1)}(z_1, z_2) = \delta_{B(\Omega_2)}(f(z_1), f(z_2))$.

Pf: First. i) is direct by Thm above.

consider approxi. of δ.

prove: $\delta_{B(\Omega_1)}(z_1, z_2) \geq \delta_{B(\Omega_2)}(\cdots)$

converse is symmetric.