$\bar{\partial}$ equations and Extension

(1) Regularity of $\bar{\partial}$ equation:

<u>Thm.</u> $\Omega \subset \mathbb{C}^n$. domain. $A(\Omega) = \{ f \in L^2_{loc}(\Omega) \mid$ weak derivative $\partial f / \partial \bar{z}_j = 0.\ \forall 1 \le j \le n \}$.

<u>Pf:</u> Consider $f_\varepsilon = f * \varphi_\varepsilon \in C^\infty(\Omega_\varepsilon)$. Smooth approxi. of f. $\Omega_\varepsilon \triangleq \{ x \in \Omega \mid d(x, \partial\Omega) > \varepsilon \}$. φ_ε is radical.

1°) Note $\partial f_\varepsilon / \partial \bar{z}_j = 0.\ \forall 1 \le j \le n$.

$\Rightarrow f_\varepsilon \in A(\Omega_\varepsilon)$.

2°) By MVT. Check: $\forall \delta, \varepsilon > 0$.

On $\Omega_{\varepsilon+\delta}$: $(f_\varepsilon)_\delta = f_\varepsilon$. $(f_\delta)_\varepsilon = (f_\varepsilon)_\delta$

$\Rightarrow f_\varepsilon = f_\delta$ on $\Omega_{\varepsilon+\delta}$

3°) Note $f_\varepsilon \xrightarrow{L^2_{loc}} f$. $\Rightarrow f_\varepsilon \overset{a.e.}{=} f$. on Ω_ε

So $f \in A(\Omega)$. (Cauchy formular holds)

<u>Rmk:</u> It means weak solution of $\bar{\partial} f = 0$ is also the strong solution

<u>Def:</u> i) ω is (p,q) form if $\omega = \sum_{\substack{\#I=p \\ \#J=q}} f_{IJ}(z)\, dz_I \wedge d\bar{z}_J$. where $I = \{ i_1 \cdots i_p \}$. $J = \{ j_1 \cdots j_q \}$. st. $i_k \ne i_t$. $j_k \ne j_t$. $i_k, j_k \le n$.

$dz_I = dz_{i_1} \wedge dz_{i_2} \cdots \wedge dz_{i_p}$. $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots d\bar{z}_{j_q}$.

ii) $C^{p,q}(\Omega) =: \{ w \text{ is } (p,q)\text{-form} \mid f_{IJ} \in C^\infty(\Omega) \}$.

$C^{(r)}(\Omega) = \bigcup_{p+q=r} C^{p,q}(\Omega)$. similar def $C_c^{p,q}(\Omega) \ldots$

iii) $\partial : C^{p,q}(\Omega) \longrightarrow C^{p+1,q}(\Omega)$ $\qquad \bar{\partial} : C^{p,q}(\Omega) \longrightarrow C^{p,q+1}(\Omega)$

$\sum_{I,J} u_{IJ} dz_I \wedge d\bar{z}_J \longmapsto \sum_{I,J} \left(\sum_{k=1}^{n} \frac{\partial u_{IJ}}{\partial z_k} dz_k \right) \wedge dz_I \wedge d\bar{z}_J$ $\qquad \sum_{I,J} u_{IJ} dz_I \wedge d\bar{z}_J \longmapsto \sum_{I,J} \left(\sum_{1}^{n} \frac{\partial u_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \right) \wedge dz_I \wedge d\bar{z}_J$

<u>Rmk</u>: So: $d = \partial + \bar{\partial}$. $\Rightarrow d^2 = 0$.

$\Rightarrow \partial^2 = \bar{\partial}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0$.

<u>Lemma</u> For $K \underset{cpt}{\subset} U \subset \mathbb{C}^n$. If $f \in C^\infty(U)$. Then $\exists V$ $\supset K$. open. and $g \in C^\infty(\mathbb{C}^n)$. s.t. $\partial g / \partial \bar{z}_1 = f$ on V.

<u>Pf</u>: WLOG. set $f \in C_c^\infty(U)$. otherwise. $\exists \varphi \in C_c^\infty$ $\varphi = 1$ on nbd V of K. set $f = \varphi f$.

Set $g(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta, z_2 \cdots z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}$.

$\in C^\infty(\mathbb{C}^n)$

Note: $\partial g / \partial \bar{z}_1 (z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial f(\zeta + z_1, \cdots z_n)}{\partial \bar{\zeta}} \cdot \frac{1}{\zeta} d\zeta \wedge d\bar{\zeta}$.

$= \lim_{r \to 0^+} \frac{1}{2\pi i} \int_{\mathbb{C} / \{|\zeta| < r\}} d\left(- \frac{f(z_1 + \zeta, z_2 \cdots z_n)}{\zeta} d\zeta \right)$

$\overset{\text{Stokes}}{=} \lim_{r \to 0^+} \int_{|\zeta| = r} f(z_1 + \zeta, z_2 \cdots z_n) / \zeta \cdot d\zeta$.

$= f(z)$.

Since $- f(z_1 + \zeta, \cdots z_n) / \zeta$ is smooth in $\mathbb{C} / \{|\zeta| < r\}$.

Lemma (Dolbeault)

For $\bar{\Delta}^n(0,r) \subset U \underset{\text{open}}{\subset} \mathbb{C}^n$. If $f \in C^{0,q}(U)$. for $q \geq 1$. $\bar{\partial} f = 0$. Then: $\exists\, V$ open. s.t. $\bar{\Delta}^n(0,r) \subset V \subset U$. and $g \in C^{0,q-1}(V)$. s.t. $\bar{\partial} g = f$ on V.

Pf: WLOG. Assume $f = \sum_{\substack{n \notin I \\ \# I = q-1}} V_I'(z)\, d\bar{z}_n \wedge d\bar{z}_I + \sum_{\substack{n \notin J \\ \# J = q}} V_J''(z)\, d\bar{z}_J$.

By Lemma above. $\exists\, u_I^{(1)} \in C^\infty(\mathbb{C}^n)$.

s.t. $\partial u_I^{(1)} / \partial \bar{z}_n = V_I'$. on $U_1 \supset \bar{\Delta}^n(0,r)$.

Set $u^{(1)} =: \sum_{\substack{n \notin I \\ \# I = q-1}} u_I^{(1)}\, d\bar{z}_I$.

$\Rightarrow f - \bar{\partial} u^{(1)} \in C^{0,q}(U_1)$ won't contain $d\bar{z}_n$.

Let: $f - \bar{\partial} u^{(1)} = \sum W_I'\, d\bar{z}_{n-1} \wedge d\bar{z}_I + \sum W_J''\, d\bar{z}_J$

s.t.: W_I'. W_J'' is holomorphic at z_n since

$\bar{\partial} f = 0$. $(*)$.

We process as before. $(*)$ guarantees $d\bar{z}_n$ won't show up. following.

$\Rightarrow f = \bar{\partial}\left(\sum_1^n u^{(k)}\right)$ on $\bigcap_1^n U_k$.

Rmk: $\bar{\partial} f$ is necessary. since that

$\bar{\partial}\bar{\partial} g = \bar{\partial} f = 0$.

Cor. For $f \in C^{0,q}(\Delta(0,r))$, $q \geq 1$, $\bar\partial f = 0$.

Then $\exists g \in C^{0,q-1}(\Delta(0,r))$ s.t. $\bar\partial g = f$

on $\Delta(0,s)$ where $\bar\Delta(0,s) \subset \Delta(0,r)$.

Pf: $\forall$ such s. $\exists t$. s.t. $\bar\Delta(0,s) \subset \Delta(0,t)$

$\subset \bar\Delta(0,t) \subset \Delta(0,r)$.

$\Rightarrow \exists h \in C^{0,q-1}(\Delta(0,t))$. $\bar\partial h = f$.

Set $\ell \in C_c^\infty(\Delta(0,t))$. s.t. $\ell = 1$ on $\Delta(0,s)$

We have $g = \ell h$ satisfies cor.

Thm (Dolbeault)

$\Delta(0,\vec r) \subset \mathbb{C}^n$. $0 < r_k \leq \infty$. $f \in C^{0,q}(\Delta(0,r))$. $q \geq 1$.

$\bar\partial f = 0$. Then. $\exists g \in C^{0,q-1}(\Delta(0,r))$. $\bar\partial g = f$ on

$\Delta(0,r)$.

Pf: Set $\Delta(0,r) = \bigcup \Delta(0,\ell_k)$. $\bar\Delta(0,\ell_k) \subset\subset \Delta(0,\ell_{k+1})$

i) $q \geq 2$:

We can find (f_k). s.t. $f_k \in C^{0,q-1}(U_{\bar\Delta_k})$

$\bar\partial f_k = f$ on some nbd of Δ_k and

$f_k|_{\Delta_{k-1}} = f_{k-1}$. $U_{\bar\Delta_k}$ is nbd of $\bar\Delta_k$.

By induction. $k=1$ ✓.

Set $\psi_{k+1} \in C^{0,q-1}(U_{\bar\Delta_{k+1}})$. $\bar\partial \psi_{k+1} = f$. on

nbd of $\bar\Delta_{k+1}$

$\Rightarrow \bar\partial(\psi_{k+1} - f_k) = 0$. on $\tilde U_{\bar\Delta_k}$.

$\overset{q \geq 2}{\exists}\ \eta_k \in C^{0,q-2}(\tilde U_{\bar\Delta_k})$. $\bar\partial \eta_k = \psi_{k+1} - f_k$ on U_k.

Set $e_k \in C^\infty(\mathbb{C}^n) \cap C_c(\Omega)$. $e_k = 1$ on nbd of $\bar{\Delta}_k$

$f_{k+1} =: \psi_{k+1} - \bar{\partial}(\eta_k e_k)$ is what we want

Def $\psi \in C^{0,q-1}(\Delta(0,r))$. $\psi|_{\Delta r_k} = f_k$.

ii) $q = 1$. By induction. find $(f_k) \subset C^\infty(U_{\bar{\Delta}_k})$.

s.t. $\bar{\partial} f_k = f$ on nbd of $\bar{\Delta}_k$. $\sup_{\Delta_{k-1}} |f_{k+1} - f_k| \le 2^{-k}$,

and $f_{k+1} - f_k \in A(\Delta_k)$.

Note. $\exists \alpha \in C^\infty(U_{\bar{\Delta}_{k+1}})$. $\bar{\partial}\alpha = f$ on nbd of $\bar{\Delta}_{k+1}$

$\Rightarrow \bar{\partial}(\alpha - f_k) = 0$ on Δ_k.

Choose $\beta = \sum_1^N a_k z^k$. s.t. $|\alpha - f_k - \beta| < 2^{-k}$.

Set $f_{k+1} = \alpha - \beta$ is what we need.

Then. let $g = \sum_{k \ge 0} (f_{k+1} - f_k)$. $f_0 = 0$.

Cor. If $f \in C^{0,1}(\Omega)$. $\bar{\partial} g = f$ in weak sense on Ω. $g \in L^1_{loc}(\Omega)$. and $\bar{\partial} f = 0$ on Ω. Then: $g \in C^\infty(\Omega)$.

Pf: $\exists u$. s.t. $\bar{\partial} u = f$ on nbd of $z_0 \in \Omega$. $u \in C^\infty(U(z_0))$.

$\Rightarrow \bar{\partial}(u - g) = 0$. in weak sense.

So: $u - g \in A(U(z_0))$.

Cor. $\Delta(0,\vec{r}) \subset \mathbb{C}^n$. $0 < r_k \le \infty$. If $f \in C^{p,q}(\Delta(0,\vec{r}))$ s.t. $\bar{\partial} f = 0$. $p,q \ge 1$. Then. $\exists g \in C^{p,q-1}(\Delta(0,\vec{r}))$ s.t. $\bar{\partial} g = f$ on $\Delta(0,\vec{r})$

Pf: Set $W(z) = \sum_{\#I=p, \#J=q} f_{IJ}\, dz_I \wedge d\bar{z}_J \in C^{p,q}(\Delta(0,\vec{r}))$.

$W_I(z) =: \sum_{\#J=q} f_{IJ}\, d\bar{z}_J \in C^{0,q}(\Delta(0,\vec{r}))$.

$\Rightarrow \exists\, u_I \in C^{0,q-1}(\Delta(0,\vec{r}))$. $\bar{\partial} u_I = W_I$ on $\Delta(0,\vec{r})$.

So: $W = \sum_I dz_I \wedge W_I$

$= \bar{\partial}\left((-1)^p \sum_I dz_I \wedge u_I\right)$ on $\Delta(0,\vec{r})$.

Rmk: It means on $\Delta(0,r)$, $\bar{\partial} u = f$ for given $f \in C^{p,q}(\Delta(0,r))$ always has a solution.

(2) Hartogs Extension:

Lemma: For $n \geq 2$. $\psi(z) =: \sum_i^n \psi_i(z)\, d\bar{z}_i \in C_c^{0,1}(\mathbb{C}^n)$.

If $\bar{\partial}\psi = 0$ on $\mathbb{C}^n$. Then: $\exists\, u \in C_c^\infty(\mathbb{C}^n)$.

s.t. $\bar{\partial} u = \psi$

Pf: Set $u(z) =: \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi(w, z_2 \cdots z_n)}{w - z_1}\, dw \wedge d\bar{w}$.

$\Rightarrow \frac{\partial u}{\partial \bar{z}_1} = \psi_1(z)$. by pf before.

Note: $\frac{\partial \psi_i}{\partial \bar{z}_j} = \frac{\partial \psi_j}{\partial \bar{z}_i}$. $i \neq j$. from $\bar{\partial}\psi = 0$.

So: $\frac{\partial u}{\partial \bar{z}_i} = \psi_i$. $\forall i \geq 1$. $\Rightarrow \bar{\partial} u = \psi$.

Note $\bar{\partial} u = 0$. on $\mathbb{C}^n / B_R$. Supp $\psi \subseteq B_R$.

$u \equiv 0$. when $|z_2|$ large enough. $\Rightarrow u = 0$ $\mathbb{C}^n / B^n$.

Rmk: $n=1$. the result doesn't hold.

Thm (Hartogs extension)

For $n \geq 2$. $\Omega \subset \mathbb{C}^n$. If $K \subset \Omega$. cpt set.

St. Ω / K is connected. $f \in A(\Omega / K)$.

Then. $\exists F \in A(\Omega)$. St. $F|_{\Omega/K} = f$.

Rmk: i) "$n=1$" doesn't hold. e.g. $\frac{1}{z}$ on $B(0,1)/\{0\}$.

ii) "Ω/K isn't connected." doesn't hold.

e.g. $f(z) = \begin{cases} 1. & 1<|z|<2 \\ 0. & 0<|z|<1 \end{cases}$ on

$B(0,2) / \{|z|=1\}$.

By uniqueness. f can't extend to $B(0,2)$.

Pf: 1) Set $\varphi \equiv 1$ on nbd of K. $\varphi \in C_0^\infty(\mathbb{C}^n)$.

$\text{supp}(\varphi) \subset \Omega$.

$\tilde{f}(z) = \begin{cases} 0. & z \in K \\ (1-\varphi) f. & z \in \Omega/K \end{cases} \in C^\infty(\Omega)$.

Besides, $\tilde{f}|_{\Omega/\text{supp}(\varphi)} = f|_{\Omega/\text{supp}(\varphi)}$.

2) Next, we consider to enlarge $\Omega/\text{supp}(\varphi)$:

Set $\psi(z) = \begin{cases} 0 & \mathbb{C}^n/\Omega \\ \bar{\partial}\tilde{f}. & \Omega. \end{cases} \in C_0^{0,1}(\mathbb{C}^n)$.

St. $\bar{\partial}\psi = 0$. on $\mathbb{C}^n$. $\text{supp}\,\psi \subset \text{supp}\,\varphi \subset\subset \Omega$.

By Lemma above. $\exists u \in C_c^\infty(\mathbb{C}^n)$. s.t. $\bar\partial u = \psi$.

$\Rightarrow u \in A(\mathbb{C}^n / \operatorname{supp}\psi)$. with uniqueness :

S. $u = 0$ on V. connected component of $\mathbb{C}^n/\operatorname{supp}\psi$

which is unbdd. (u is for guarantee holo-).

3) Set $F = \tilde f - u$. $\in C^\infty(\Omega)$. $\bar\partial F = 0$. on Ω.

Note $\partial V \subset \operatorname{supp}(\psi)$. $\Rightarrow V \cap \Omega \neq \emptyset$.

$F|_{\Omega \cap V} = (\tilde f - u)|_{\Omega\cap V} = \tilde f|_{\Omega \cap V} = f$.

By uniqueness Thm. $\Rightarrow F = f$ on Ω / K

Thm. $\Omega \subset \mathbb{C}^n$ domain. s.t. $\forall f$ given. $f \in C^{0,1}(\Omega)$. $\Rightarrow$.

$\bar\partial w = f$ has solution. Then $\forall u \in A(\Omega \cap \{z_1 = 0\})$

can be extended to $\tilde u \in A(\Omega)$. $\tilde u|_{\Omega\cap\{z_1=0\}} = u$.

Rmk: We can criteria whether a domain

satisfies "Dolbeault Thm" by finding

whether $\exists u$ can't extend from $\Omega \cap \{z_1 = 0\}$.

Cor. For $\Omega \subseteq \mathbb{C}^n$. domain. $f \in A(\Omega) \cap C(\bar\Omega)$.

If $f(a) = 0$. for some $a \in \Omega$. Then:

$\exists b \in \partial\Omega$. s.t. $f(b) = 0$

Pf: If $f \neq 0$ on $\partial\Omega$. Set $\Omega_\delta =: \{x \in \Omega \mid d(x, \partial\Omega) > \delta\}$.

$\Rightarrow \exists\, \delta > 0$. s.t. $f \neq 0$ on $\Omega / \bar\Omega_\delta$ by conti.

So: $1/f \in A(\Omega)$ on $\Omega / \bar{\Omega}_0$

extend to F on Ω. s.t. $F^{-1}|_{\square} = f$.

$\overset{\text{unique}}{\Rightarrow}$ $f = F^{-1}$ on Ω. $\therefore$ $f \neq 0$ on Ω.

<u>Cor.</u> $\forall f \in A(\Omega)$. can't have isolated zero.

<u>Pf:</u> Process as above. By contradiction:

Set U is nbd of such zero.

extend f^{-1} from $\Omega / \bar{U}$ to Ω.

<u>Cor.</u> For $K \underset{cpt}{\subset} \mathbb{C}^n$. if $\mathbb{C}^n / K$ is connected.

$f \in A(\mathbb{C}^n / K)$ is bdd. Then $f \equiv$ const.

<u>Pf:</u> Exten f on $\mathbb{C}^n$. still bdd.

By Cauchy estimate $\Rightarrow f^{(n)} \equiv 0$.

<u>Cor.</u> $f \in A(\mathbb{C}^n)$. If $Z(f) \neq 0$. Then $Z(f)$ is unbdd set.

<u>Pf:</u> Similarly. Cover $Z(f)$ by B_R.

if it's bdd. by contradiction.

(3) <u>Bochner - Martinelli Formula:</u>

First, we want to define differentiation of functions on submanifolds $\subset \mathbb{C}^n$.

For $U \subset \mathbb{C}^n$. domain. recall that for $S \subset U$. closed. C^k- submanifold with $\dim = 2n-1$. satisfies:

$\forall p \in S$. $\exists U_p$. $\varphi \in C^k(U_p, \mathbb{R}^1)$. St. $S \cap U_p = \{z \in U_p \mid \varphi = 0\}$.

and $r(\nabla \varphi) = 1$. on U_p.

We call such kind φ by definition func. at p.

<u>Lemma.</u> If $\varphi_1 \in C^k(U_p, \mathbb{R}^1)$. is definition func. at p of S. $\varphi_2 \in C^k(U_p, \mathbb{R}^1)$ St. $\varphi_2 \equiv 0$ on $U_p \cap S$. Then. $\exists h \in C^{k-1}(U_p, \mathbb{R}^1)$. St.

i) $\varphi_2 = h\varphi_1$ on U_p. ii) $d\varphi_2 = h\,d\varphi_1$ on $U_p \cap S$.

<u>Pf:</u> WLOG. Set $p = 0$. $\varphi_1(z) = y_n$ and.

$U_p \cap S = \{z \in U_p \mid y_n = 0\}$. convex.

(Otherwise set $\Psi : (x_1, y_1, \cdots x_n, y_n) \xrightarrow[C^k]{\sim} (x_1, y_1, \cdots x_n, \varphi_1(z))$. compose with φ_1)

$\Rightarrow \varphi_2(\tilde{z}, x_n, 0) = 0$ on U_p.

So: $\varphi_2(\tilde{z}, x_n, y_n) = y_n \int_0^1 \frac{\partial \varphi_2}{\partial y_n}(\tilde{z}, x_n, t y_n) dt$.

set $h(z) = \int_0^1 \frac{\partial \varphi_2}{\partial y_n}(\tilde{z}, x_n, t y_n) dt \in C^{k-1}$.

<u>Rmk:</u> For φ_1, φ_2 complex-valued. we can separate their real and imagined parts. $\Rightarrow \exists h \in C^{k-1}(U_p, \mathbb{C})$.

Def: i) For $\varphi \in C^k(U_p)$ is definition func. of S at p. Set:

$T_p(S) =: \{T = \sum t_i \frac{\partial}{\partial x_i} + s_i \frac{\partial}{\partial y_i} \mid T(\varphi)|_p = 0.\ t_i.$

$s_i \in \mathbb{R}^1\}$. tangent space of S.

$T_p^{0,1}(S) =: \{T = \sum s_i \frac{\partial}{\partial \bar{z}_i} \mid T(\varphi)|_p = 0.\ s_i \in \mathbb{C}^1\}$.

anti-holo tangent space.

Rmk: i) By Lemma above. T_p. $T_p^{0,1}$ will

be indept of choice of φ.

ii) $T_p(S)$ is real LS with dim $= 2n-1$.

$T_p^{0,1}(S)$ is complex LS with dim $= n-1$

ii) $\forall f \in C^1(S)$. $p \in S$. $\Rightarrow \exists U_p$ and $F \in C^1(U_p)$

s.t. $F|_{U_p \cap S} = f|_{U_p \cap S}$. We set:

$T(f)(p) := T(F)(p)$. $\forall T \in T_p^{0,1}(S)$.

Rmk: It's well-defined since by Lemma:

For $h \in C^1(U_p)$ another extension.

$\Rightarrow (F - h)|_{U_p \cap S} = 0$. So $\exists h \in C(U_p)$

s.t. $F - h = h\varphi$. Then:

$T(F-h)(p) = h(p)\, T(\varphi)(p) = 0$.

iii) $f \in C^1(S)$ satisfies tangent Cauchy-Riemann

equation if $T(f)(p) = 0$. $\forall T \in T_p^{0,1}(S)$. $\forall p \in S$.

Rmrk: For $n>1$. $\Omega\subset\mathbb{C}^n$. odd domain and $\partial\Omega\in C^\infty$. $f\in C^1(\partial\Omega)$. If

$$\begin{cases}\bar\partial\varphi=0 & \text{in } \Omega\\ \varphi=f & \text{on } \partial\Omega\end{cases}\quad \text{has solution } \varphi$$

s.t. $\varphi\in C^1(\bar\Omega)$. $\Rightarrow\ \exists\,\tilde\varphi$ extend φ on $\mathbb{C}^n$.

So: $\bar\partial f=0\ \Rightarrow\ Tf\equiv 0$. on $S=\partial\Omega$.

Lemma. S has definition function $\varphi\in C^\infty(\Omega,\mathbb{R}^1)$ in Ω. s.t. $d\varphi\neq 0$. on Ω. For $f\in C^1(\Omega,\mathbb{C}^1)$. we have:

i) f satisfies tangent C-R equation

ii) $\bar\partial f\wedge\bar\partial\varphi=0$ on S.

iii) $df\wedge(dz_1\wedge\cdots dz_n)=0$ on $T_p(S)$. $\forall p\in S$.

st. i). ii). iii) are equi.

__Pf:__ WLOG. fix $p=0\in S$. $\varphi=\lambda y_n$. $\lambda\in\mathbb{C}^1$.

and $T_p(S)=\{\operatorname{Im} z_n=0\}$.

Since $d\varphi(p)=\frac{i}{2}\lambda(dz_n+d\bar z_n)$.

So: $T_p^{0,1}(S)=\operatorname{span}\{\frac{\partial}{\partial\bar z_1}\cdots\frac{\partial}{\partial\bar z_{n}}\}$.

i) $\Leftrightarrow$ $\frac{\partial f}{\partial\bar z_k}=0$. $\forall\ 1\le k\le n-1$.

Besides. $df\wedge(\cdots)=\sum_1^{n?}\frac{df}{\partial\bar z_k}(p)\,d\bar z_k\wedge(\cdots)$

since $d\bar z_n=dz_n$. on $T_p(S)$.

Def: i) For $\alpha = \sum_{\substack{\#I=s \\ \#J=r}} \alpha_{IJ}\, dz_I \wedge d\bar{z}_J$, $t = \sum t_i \frac{\partial}{\partial x_i} + s_i \frac{\partial}{\partial y_i}$

$$t \lrcorner \alpha := \sum (-1)^{\ell-1} \alpha_{IJ}\, t_{i_\ell}\, dz_{i_1} \wedge \cdots \widehat{dz_{i_\ell}} \wedge \cdots \wedge d\bar{z}_J$$

$$+ \sum (-1)^{s+\ell-1} \alpha_{IJ}\, s_{j_\ell}\, dz_I \wedge d\bar{z}_{j_1} \wedge \cdots \widehat{d\bar{z}_{j_\ell}} \wedge \cdots$$

ii) Set $e := dz \wedge d\bar{z} / |z|^{2n}$. $\bar{e} := \sum \bar{z}_j \frac{\partial}{\partial \bar{z}_j}$

$$W := \frac{(n-1)!}{(2\pi i)^n}\, \bar{e} \lrcorner e \quad . \quad W_\zeta(z) := W(z-\zeta).$$

Lemma i) $n=1$. $W_\zeta(z) = \frac{1}{2\pi i} \cdot (z-\zeta)^{-1} dz$.

ii) $dW_\zeta \equiv 0$ on $\mathbb{C}^n / \{\zeta\}$.

iii) $\int_{|z-\zeta|=r} W_\zeta(z) = 1$.

iv) $U \underset{\text{bdd}}{\subset} \mathbb{C}^n$, $\partial U \in C^\infty$. $\zeta \in U$. Then: $\int_{\partial U} W_\zeta(z) = 1$.

v) $G(\zeta) =: \int_{\partial U} f(z) W_\zeta(z)$. if $f \in C(\partial U)$.

and $U \subset \mathbb{C}^n$ bdd with $\partial U \in C^\infty$. Then:

$G(\zeta)$ is harmonic on $\mathbb{C}^n / \partial U$.

vi) $\frac{\partial W_\zeta(z)}{\partial \bar{\zeta}_j} = d\left(-\frac{\partial}{\partial \bar{z}_j} \lrcorner W_\zeta(z)\right)$. $\forall j$ on $\mathbb{C}^n/\{\zeta\}$

$W_\zeta(z) = d\left(-\frac{|z-\zeta|^2}{(n-1)(z_j-\zeta_j)} \frac{\partial}{\partial \bar{z}_j} \lrcorner W_\zeta(z)\right)$ on

$\mathbb{C}^n / \{z_j = \zeta_j\}$. $\forall j$.

Pf: i), ii), vi) are easy to check.

Apply Stokes Thm. on iii).

iv). Consider $B(\zeta, r) \subset U$. $\int_{\partial B(\zeta,r)} = \int_{\partial U}$

follow from Stokes.

v) Note : $\dfrac{\bar{z}_j - \bar{\zeta}_j}{|z-\zeta|^{2n}} = \dfrac{\partial}{\partial z_j} \left((1-n)^{-1} |z-\zeta|^{2(1-n)} \right)$

Rmk: $G(\zeta)$ may not be holo- when $n>1$

Thm (Bochner - Martinelli)

$$f(z) = \int_{\partial U} f(\zeta) W_z(\zeta) - \int_U \bar\partial f(\zeta) \wedge W_z(\zeta)$$

for $f \in C^1(\bar{U})$ where $U \underset{bdd}{\subset} \mathbb{C}^n$. ∂U piecewise smooth.

Pf: $d(f(z) W_\zeta(z)) = df \wedge W_\zeta = \bar\partial f \wedge W_\zeta$

on $U / B(\zeta, r)$ for $B(\zeta,r) \subset U$.

$$\int_{\partial U} f(z) W_\zeta(z) \overset{\text{stokes}}{=} \int_{\partial B(\zeta,r)} f W_\zeta - \int_{U/B(\zeta,r)} d(f W_\zeta).$$

$$= \int_{\partial B(\zeta,r)} f W_\zeta - \int_{U/\square} \bar\partial f \wedge W_\zeta$$

$$\xrightarrow{r \to 0} f(\zeta) - \int_U \bar\partial f \wedge W_\zeta(z).$$

Rmk: It's extension of Cauchy formula in $n=1$.

Cor. $U \underset{bdd}{\subset} \mathbb{C}^n$. piecewise smooth. $f \in A(U) \cap C^1(\bar{U})$.

$$\Rightarrow \int_{\partial U} f(z) W_\zeta(z) = f(\zeta).$$

cor. $\int_{\partial U} W_\zeta(z) = \tilde{\chi}_U \triangleq \begin{cases} 1 . & z \in U \\ 0 . & z \notin \bar{U} \end{cases}$

(4) <u>Bochner - Severi Extension:</u>

<u>Thm.</u> For $n \geq 2$. $U \underset{bdd}{\subset} \mathbb{C}^n$ with $\partial U \in C^1$. $\mathbb{C}^n / \bar{U}$ connected.

If $f \in C^1(\partial U)$. satisfies tangent C-R equation on ∂U

Then $F(\zeta) = \int_{\partial U} f(z) W_\zeta(z)$ satisfies:

i) $F \in A(\mathbb{C}^n / \partial U)$ ii) $F \equiv 0$ on $\mathbb{C}^n / \bar{U}$

iii) $F \in A(U)$ can extend to ∂U conti. st. $\bar{F}|_{\partial U} = f$.

<u>Pf:</u> i) $\forall \zeta \notin \partial U$. $\dfrac{\partial F(\zeta)}{\partial \bar{\zeta}_j} = \int_{\partial U} f(z) \dfrac{\partial W_\zeta(z)}{\partial \bar{\zeta}_j}$

$$\overset{\text{Lem.}}{=} - \int_{\partial U} f(z) \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner W_\zeta(z) \right)$$

Note $\int_{\partial U} \partial f(z) \wedge \left(\frac{\partial}{\partial \bar{z}_j} \lrcorner W_\zeta(z) \right) = 0$.

since f satisfies tangent C-R equation

So: RHS $= - \int_{\partial U} d\left(f(z) \frac{\partial}{\partial \bar{z}_j} \lrcorner W_\zeta(z) \right)$

$$\overset{\text{stokes}}{=} 0 \quad . \quad \partial(\partial U) = \emptyset .$$

ii) Set $V := \{ \zeta \in \mathbb{C}^n / \bar{U} \mid |\zeta_1| > \max_{\bar{U}} |z_1| \}$.

is connected domain $\subset \mathbb{C}^n / \bar{U}$.

Set $t_1 := -|z-\zeta|^2 / (n-1)(z_1 - \zeta_1) \cdot \frac{\partial}{\partial \bar{z}_1}$ in V

$\Rightarrow F(\zeta) = \int_{\partial U} f(z) \, d(t_1 \lrcorner W_\zeta(z)) = 0$

Similarly as above.

with uniqueness Thm. $F(\zeta) = 0$ on $\mathbb{C}^n/\bar{\Omega}$.

iii) $\forall \zeta_0 \in \partial\Omega$. recall. $\int_{\partial\Omega} W_\zeta(z) = \tilde{\chi}_\Omega \triangleq \begin{cases} 1 . & \Omega \\ 0 . & \bar{\Omega}^c \end{cases}$

Set $h(\zeta) := F(\zeta) - f(\zeta_0)\tilde{\chi}_\Omega$

$$= \int_{\partial\Omega} (f(z) - f(\zeta_0)) W_\zeta(z) . \ \zeta \notin \partial\Omega.$$

Note $(f(z) - f(\zeta_0)) W_\zeta(z) = O(|z-\zeta_0|^{-2n+2} \wedge S(z))$

and $\dim(\partial\Omega) = 2n-1$. $\Rightarrow h(\zeta_0)$ converges.

$h(\zeta) - h(\zeta_0) = \int_{\partial\Omega \cap V(\zeta_0)} + \int_{\partial\Omega / V(\zeta_0)} \xrightarrow{\zeta \to \zeta_0} \int_{\partial\Omega \cap V(\zeta_0)} \leq \varepsilon$.

since the second term is conti. and we choose $V(\zeta_0)$ small and st. $\int_{\partial\Omega \cap V(\zeta_0)} \leq \varepsilon$.

So: $\lim_{\zeta \to \zeta_0} h(\zeta) = h(\zeta_0)$. i.e. $\lim_{\zeta \to \zeta_0} F(\zeta) = f(\zeta_0)$

<u>Rmk</u>: If $\mathbb{C}^n/\bar{\Omega}$ isn't connected. e.g. $f = \begin{cases} 1 . & |z|=1 \\ 2 . & |z|=2 \end{cases}$

for $\Omega = B(0,2)/\bar{B}(0,1)$. $\Rightarrow F(\zeta) = 1$ on Ω.

contradict with continuity

<u>Thm.</u> For $n \geq 2$. $D \subset \mathbb{C}^n$. domain. $S \subset D$ is smooth real surface with dim $2n-1$. Then: we have,

$A(D/S) \cap C(D) = A(D)$.

<u>Rmk</u>: When $n=1$. it also holds. follows directly by Morera Thm.