Holomorphic sets

Next. we will investigate the zeros of high dimensional holomorphic functions.

(1) <u>Commutative Algebra:</u>

Next. we consider R is commutative ring with id.

<u>Def</u>: R is Noetherian ring if $\forall (I_n)$ seq of ideals. st. $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$. $\exists N$. st. $I_N = I_m$. $\forall m \geq N$.

<u>Lemma.</u> R is Noetherian $\Leftrightarrow$ Any ideal I of R. is finitely generated. if $|I| = S$

<u>Pf</u>: $(\Rightarrow)$ Let $I_0 = (a_0)$. $a_0 \in I$.

$I_1 = (a_0, a_1)$. $a_1 \in I / (a_0)$.

$\cdots$ $I_0 \subset I_1 \subset \cdots \subset I = \bigcup (a_i)$.

So: $I = I_N$ for some N.

$(\Leftarrow)$ Similarly. for seq $I_0 \subset I_1 \subset \cdots$

Let $I = \bigcup I_k = [a_0, \cdots a_N]$. $\supset I_k$. $\forall k$.

So $\forall a_n$. $\exists N_n$. st. $a_n \in I_{N_n}$

$\Rightarrow \exists \tilde{N} \geq \max\limits_{n} N_n$. $I = I_{\tilde{N}}$.

Lemma R is Noetherian $\iff$ $\forall$ submodule of finite generated R-module is also finitely generated.

Rmk: So: $\forall$ field and PID are Noetherian.

Thm (Hilbert)

R is Noetherian $\Rightarrow$ $R[x_1 \cdots x_n]$ is Noetherian.

(2) Weierstrass Thm:

Thm If $f(z) \in A(U_a)$. $f(a)=0$. $f(\tilde{a}, z_n) \not\equiv 0$

Then. $\exists$ nbd $\tilde{V} \times V_n$ of $\tilde{a} \times a_n$. s.t.

$$f(z) = \left((z_n - a_n)^k + \sum_{i=1}^{k} c_i(\tilde{z})(z_n - a_n)^{k-i}\right) \varphi(z)$$

where k is degree of zero $z_n = a_n$. and

$c_i(\tilde{a}) = 0$. $c_i \in A(V')$. $\varphi \in A(\tilde{V} \times V_n)$ has no zeros.

Pf: WLOG. set $a=0$. $f(\tilde{0}, a_n) \not\equiv 0$.

conti $\Rightarrow$ $\exists r_n > 0$. s.t. $f(\tilde{0}, z_n) \neq 0$ on $|z_n| \leq a_n$.

$\exists V(\tilde{0}, r)$. s.t. $f(\tilde{z}, z_n) \neq 0$. on $V(\tilde{0}, r) \times \{|z_n| = r_n\}$

Fix $\tilde{z}$. set $(z_n(\tilde{z}))_1^k$ is zeros of $f(\tilde{z}, z_n)$ in V_n. $p(z) = \prod_1^k (z_n - z_n(\tilde{z}))$

$$= z_n^k + c_1(\tilde{z}) z_n^{k-1} + \cdots + c_k(\tilde{z}).$$

$p(\tilde{0}, z_n) = z_n^k \Rightarrow c_i(\tilde{0}) = 0$. $\forall i \leq k$.

1) Prove: $c_i(\tilde{z})$ is holomorphic.

<u>Lemma.</u> $f \in A(D)$, $D \subset \mathbb{C}^1$ has zeros (z_k)

$$\Rightarrow \sum_{k=1}^{n} z_k^m = \frac{1}{2\pi i} \oint_{\partial B_r} \zeta^m \partial_\zeta f \, d\zeta / f(\zeta)$$

where $(z_k)_1^n \subset B_r$

So: $\sum_1^k (z_n(\tilde{z}))^m = \frac{1}{2\pi i} \int_{|z_n| = r_n} (z_n)^m \frac{\partial_{z_n} f(\tilde{z}, z_n) dz_n}{f(\tilde{z}, z_n)}$

$\Rightarrow$ LHS is holomorphic for $\forall m \geq 1$.

So: $c_i(\tilde{z})$ is holomorphic since it's sym poly which can be represented by them

2) Set $\varphi(z) = f(z)/p(z) \in A(\{|z_n| \leq r_n\})$

has no zeros in $|z_n| \leq r_n$. $\forall \tilde{z}$ fixed.

$\Rightarrow \varphi(\tilde{z}, z_n) = \frac{1}{2\pi i} \int_{|\zeta| = r_n} \frac{(f/p)(\tilde{z}, \zeta)}{\zeta - z_n} d\zeta$

So: $\varphi \in A(V(\tilde{o}, r) \times V_n)$ follows from

$p \in A(V(\tilde{o}, r) \times V_n)$ has no zeros.

<u>Def:</u> Weierstrass polynomial with degree $= k$ at a

w.r.t z_n has form: $(z_n - a_n)^k + \sum_{i=1}^k c_i(\tilde{z})$

$(z_n - a_n)^{k-i}$. where $c_i(\tilde{z}) \in A(U_{\tilde{a}})$.

<u>Rmk:</u> By Thm above. to investigate the

zeros of f at $z = a$. we only need

to consider its Weierstrass polynomial.

Thm. (Weierstrass Division)

For $a \in \mathbb{C}^n$, $f \in A(U_a)$. P is Weierstrass polynomial with degree k at a w.r.t z_n.

Then $\exists V_a \subset U_a$ s.t. $f \underset{\text{uni que}}{=} \varphi p + a$. $\varphi \in A(V_a)$

and a is polynomial w.r.t z_n with degree $\leq k-1$ and has coefficients $\in A(\widetilde{V}_{\widetilde{a}})$.

Pf: WLOG. set $a = 0$.

1) Existence:

As above. $\exists \{|z_n| = r_n\}$. $V(\widetilde{0}, r)$. s.t.

$p(\widetilde{z}, z_n) \neq 0$. $\forall |z_n| = r_n$, $\widetilde{z} \in V(\widetilde{0}, r)$.

Set $\varphi(z) = \frac{1}{2\pi i} \int_{|\zeta| = r_n} \frac{(f/p)(\widetilde{z}, \zeta)}{\zeta - z_n} d\zeta$.

$\Rightarrow a(z) = f - p\varphi = \frac{1}{2\pi i} \int_{|\zeta| = r_n} \frac{f(p(\widetilde{z}, \zeta) - p(\widetilde{z}, z_n))}{p(\widetilde{z}, \zeta)(\zeta - z_n)}$

which is degree $\leq k-1$. w.r.t. z_n.

and has holomorphic coefficient.

2) Uniqueness:

$\exists \{|z_n| < r_n\}$. $V(\widetilde{0}, r)$. s.t.

$p(\widetilde{0}, z_n) \neq 0$. $\forall 0 < |z_n| \leq r_n$. and that

$|p(\widetilde{z}, z_n) - p(\widetilde{0}, z_n)| \leq \frac{1}{2} \sup_{|z_n| = r_n} |p(\widetilde{0}, z_n)|$

for $\forall \widetilde{z} \in V(\widetilde{0}, r)$.

By Rouché. $\Rightarrow p(\widetilde{z}, z_n)$ also has

k zeros in $\{|z_n| < r_n\}$. as $p(\widetilde{0}, z_n)$

If $f = \varphi_1 p + a_1 = \varphi_2 p + a_2$. Fix $\tilde{z} \in V(\tilde{0}, r)$.

Then $p(\varphi_1 - \varphi_2) = a_1 - a_2$. but LHS has at least k zeros in $\{|z_n| < r_n\}$ while RHS has at most $k-1$ zeros.

$\Rightarrow \varphi_1 = \varphi_2$. $a_1 = a_2$ in $\{|z_n| < r_n\} \times V(\tilde{0}, r)$.

<u>Def</u>: $\mathcal{O}_a^n := \{ \sum_k a_k (z-a)^k$ converges in nbd of $a \in \mathbb{C}^n \}$

<u>Thm.</u> $\mathcal{O}_a^n$ is Noetherian ring.

<u>Pf</u>: WLOG. set $a = 0$. Induction on n.

$\mathcal{O}_0^0 \cong \mathbb{C}^n$ holds. For $\mathcal{O}_0^n$:

$\forall I \subsetneq \mathcal{O}_0^n$. ideal. $\exists f \in I$. $f(0) = 0$.

$f = ph$. where p is k-Wierstrass poly and h has no zeros in $\mathcal{O}_a^n$.

Apply Weierstrass Division on $\forall g \in I$.

$g = p\varphi + r$. $\Rightarrow$ By hypo. All the r consist a submodule N. So: N is Noetherian

$N := \langle r_1 \cdots r_k \rangle$. $\Rightarrow I = \langle f, r_1 \cdots r_k \rangle$.

<u>Rmk</u>: $A(\mathbb{C}')$ isn't Noetherian.

Set $J_k := \{ f \in A(\mathbb{C}') .\ f(m) = 0.\ \forall m \geq k.\ m \in \mathbb{Z} \}$

Seq of ideal. But $J_k / J_{k+1} \neq 0$. $\forall k \geq 1$.

(3) <u>Holomorphic Sets:</u>

<u>Def</u>: For domain $D \subset \mathbb{C}^n$.

i) A is holomorphic set in D if $A \underset{\text{close}}{\subset} D$

and $\forall z_0 \in A$. $\exists U(z_0) \subset D$. $g_i \in A(U(z_0))$. $i \le k$.

st. $A \cap U(z_0) = \{ z \in U(z_0) \mid g_1(z) = \cdots g_k(z) = 0 \}$.

<u>Rmk</u>: It's equi. with definition:

$\forall z_0 \in D$. $\exists U(z_0) \subset D$. $g_i \in A(U(z_0))$. $i \le k$.

st. $A \cap U(z_0) = \{ z \in U(z_0) \mid g_1(z) = \cdots g_k(z) = 0 \}$.

<u>Pf</u>: ($\Rightarrow$) $\forall z_0 \in D/A$. $\exists U(z_0) \subset D$. $U(z_0) \cap A = \emptyset$.

and $g \equiv c \neq 0$. So $\{ g = 0 \} = \emptyset = \text{LHS}$.

($\Leftarrow$). prove: A is close in D.

if $\exists z_0 \in D/A$. st. $\forall$ nbd of z_0

$\cap A \neq \emptyset$. By def in Rmk:

$\exists U(z_0)$. $g_i \in A(U(z_0))$ $A \cap U(z_0) = \cdots$

But $z_0 \notin A$. so by conti. of g_i.

$\exists V(z_0)$. st. $g_i(z) \neq 0$ on $V(z_0)$. $\forall i \le k$.

$\Rightarrow$ $A \cap V(z_0) = \emptyset$. Contradict!

ii) For A holomorphic in D. $A \neq \emptyset$. it's principle

holomorphic if $\exists g \not\equiv 0 \in A(D)$. st. $A = \{ z \in D \mid g = 0 \}$.

it's locally principle holo. if $\forall z_0 \in A$. $\exists U(z_0)$

st. $A \cap U(z_0)$ is principle holomorphic in $U(z_0)$

Rmk: i) $\emptyset$, D are holo in D. ($g=0$; $g=1$)

ii) Submanifold in D is holomorphic in D.

iii) A_1, A_2 holomorphic in D. Then

$A_1 \cap A_2$. $A_1 \cup A_2$. $A_1 \times A_2$ are holomorphic.

$F: D \xrightarrow{\sim} F(D)$. biholo. $\Rightarrow F(A) \underset{\text{holo.}}{\subset} F(D)$.

Thm A is holomorphic in $D \Rightarrow A$ is nowhere dense.

Pf: Prove: $\operatorname{int} \bar{A} = \operatorname{int} A = \emptyset$.

By contrad.: if z_0 is limit point in $\operatorname{int} A$.

Since $\exists U(z_0) \subset D$. g_i. $i \le k$. st. $g_i \in A(U(z_0))$.

$U(z_0) \cap A = \{ z \in U(z_0) \mid g_i = 0.\ i \le k \}$ contains

a open set in D. $\Rightarrow g_i \equiv 0$ on $U(z_0)$.

So: $U(z_0) \cap A = U(z_0)$. $z_0 \in \operatorname{int} A$.

$\Rightarrow$ A is clopen. i.e. $A = \emptyset$ or D.

Rmk: Easy to see: if $n=1$. $A \subset \mathbb{C}^1$. then.

$A = \emptyset$. D. or discrete set.

Def: A is holomorphic in D. $z_0 \in A$. is called regular point if $\exists V(z_0) \subset D$. st. $A \cap V(z_0)$ is submanifold in $V(z_0)$. Denote $A_{reg} \subset A$ is set of such points.

$A_{sing} =: A / A_{reg}$. set of singular points.

Rmk: When $A_{sing} = \emptyset$ $\Rightarrow$ A is submanifold in D.

Thm A_{reg} is open complex submanifold in A.

Pf: $A \cap V(z_0) = \{ z \in V(z_0) \mid f_1 = \cdots f_k = 0 \}$. $z_0 \in A_{reg}$.

Note that $\left(\frac{\partial f_k}{\partial z_j}\right)(z_0)$ has rank k.

$\Rightarrow$ it also holds in some nbd $U(z_0)$ of z_0.

Thm A_{reg} is open dense set in A. So: A_{sing} is nowhere dense closed in A.

Pf: Apply induction on dimension n.

$n=1$. $A = \emptyset$. D. or discrete set $\Rightarrow$ $A = A_{reg}$.

if $n=k$. $\forall a \in A$. $\forall$ nbd U of a. WLOG. set:

$A \cap U = \{ z \in U \mid f_1 = \cdots f_N = 0 \}$. prove: $A_{reg} \cap U \neq \emptyset$.

WLOG. $\exists i$. $f_i \not\equiv 0$. otherwise it's trivial. set $f_1 \not\equiv 0$.

By Thm before. $\exists V \subset U$. st. $Z(f_1, V)$ is $(k-1)$ dimensional manifold. $A \cap V \subset Z(f_1, V)$.

By inductive hypothesis $A_{reg} \cap V \neq \emptyset$.

Thm. A_{sing} belongs to another holomorphic set different from A.

Pf: WLOG. Set $A \neq \emptyset$ or D. For $a \in A_{sing}$

Set $k := \max\{N \in \mathbb{Z}^+ \mid \exists U(a).\ f_i \in A(U(a))$

$f_i \equiv 0$ and $|D| = |(\frac{\partial f_i}{\partial z_{ji}})_{N\times N}| \neq 0\}$. $i \leq N$. on $A \cap U(a)\}$

Set $V := \{z \in U(a) \mid |D(z)| \neq 0\}$.

$$\tilde{A} := \{z \in U(a) \mid f_1 = \cdots f_k = 0\} \cap V.$$

$\Rightarrow \tilde{A}$ is submanifold in V. $A \cap V \subseteq \tilde{A}$.

Note $\forall f \in A(U(a))$. $f = 0$. df must be LS of df_i. $i \leq k$. otherwise add f into $(f_i)_1^k$. $\Rightarrow k$ isn't max.

So: $d(f|_{\tilde{A}}) \equiv 0$. $f|_{\tilde{A}} \equiv 0$. $\Rightarrow \tilde{A} = A \cap V$.

$\Rightarrow A \cap V = \tilde{A} \cap A_{reg} = A_{reg} \cap V \Rightarrow A_{sing} \cap V = \emptyset$.

So: $A_{sing} \cap U \subset A \cap \{z \in U \mid D(z) = 0\}$.

$$= \{z \in U \mid f_1 = \cdots f_k = D(z) = 0\}$$

Def: i) For $p \in A$. dimension of A at p is:

$$\dim_p A = \begin{cases} \dim_p A \cap V(p). & p \in A_{reg} \\ \varlimsup\limits_{\substack{p' \in A_{reg} \\ \to p}} \dim_{p'} A \cap V(p') & p \in A_{sing} \end{cases}$$

Set $\dim A = \sup\limits_{p \in A} \dim_p A$.

Rmk: i) $\dim_p A \in [0, n]$. is u.s.c.

ii) By density. $\dim A = \sup\limits_{A_{reg}} \dim_p A$

iii) By Thm above. A_{sing} belongs to a holomorphic set with $\dim < \dim A$.

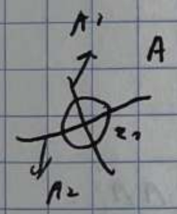
ii) $A, B \subset D$. $z_0 \in D$. $A \sim B$ at z_0 if $\exists V(z_0) \subset D$. st. $A \cap V(z_0) = B \cap V(z_0)$.

Denote the equi. class of A at z_0 by (A, z_0) called sheaf of A at z_0.

iii) (A, z_0) is a sheaf in D. Set:

$J_{(A,z_0)} := \{ f \in \hat{\mathcal{O}}_{z_0} \mid \exists V(z_0) \subset D.\ st.\ f|_{A \cap V(z_0)} = 0 \}$.

ii) A is holomorphic. we say it's irreducible if $(A, z) = (A_1, z) \cup (A_2, z)$. A_i are holo $\Rightarrow \exists i.\ A_i = A$.



<u>Rmk:</u> J_{A,z_0} is a ideal in $\hat{\mathcal{O}}_{z_0}$.

<u>Thm</u> (A, z) is irred. $\Leftrightarrow$ $J_{(A,z)}$ is prime ideal of $\hat{\mathcal{O}}_z$.

<u>Pf:</u> $(\Rightarrow)$. $fg \in J_{(A,z)} \Rightarrow (A, z) = (A \cap f^{-1}(0), z) \cup (A \cap g^{-1}(0), z)$

since $\{fg = 0\} = f^{-1}(0) \cup g^{-1}(0)$.

By irred. $\exists f \in J_{A,z}$. i.e. $A \cap f^{-1}(0) = A$.

$(\Leftarrow)$ If $(A, z) = (A_1, z) \cup (A_2, z)$. $A_i \neq A$.

$\Rightarrow \exists f \in J_{A_1,z}$, $g \in J_{A_2,z}$. $fg \in J_{A,z}$

But $f, g \notin J_{A,z}$. Contradict!

Thm. We can't have $(A_n)_1^\infty$ sheafs at z_0. s.t.

$(A_1, z_0) \supsetneq (A_2, z_0) \supsetneq (A_3, z_0) \cdots$

Pf: If so. $\Rightarrow J_{A_1, z_0} \subsetneq J_{A_2, z_0} \subsetneq \cdots$

But $\mathcal{O}_a^n$ is Noetherian. $\Rightarrow$ contradict!

Thm. $\forall (A, z_0)$ sheaf has a unique finite decomposition : $(A, z_0) = \bigcup_1^N (A_k, z_0)$. where (A_k, z_0). $k \le N$ are irred. sheafs.

Pf: By Thm above. the decomp. is finite.

If $(A, z) = \bigcup_1^N (A_i, z) = \bigcup_1^N (A_i', z)$.

Then : $(A, z) = \bigcup (A_i \cap A_j', z)$.

By irred. $A_i \subset A_i \cap A_j'$. $A_j' \subset A_i \cap A_j'$.

Thm For $\mathcal{J} \subset A(D)$. $A = \{ f = 0.\ f \in \mathcal{J} \} \cap D$ is holomorphic in D.

Pf: For $z_1 \in A$. Note $J_{A, z_1} \subset \mathcal{O}_{z_1}^n$ Noetherian :

$J_{A, z_1} = \langle f_1, \cdots f_n \rangle$. Set $V(z_1)$ nbd.

s.t. $f_k \in A(V(z_1))$. $f_k|_{A \cap V(z_1)} = 0$.

Def : $\widetilde{A} = \{ z \in V(z_1) \mid f_1(z) = \cdots f_n(z) = 0 \}$

Easy to see $(\widetilde{A}, z_0) = (A, z_0)$.

Thm (Riemann Extension)

$S \subsetneq D$. holomorphic. If $f \in A(D/S)$. and local bdd in nbd of S. Then f can be holomorphically extend to D. uniquely.

Pf: 1) Note $D/S \underset{\text{dense}}{\subset} D$. $\Rightarrow$ So it's unique.

2) Note the conclusion is local.

WLOG. set $S = \{z \in D \mid f_1 = \cdots f_m = 0.\ f_k \in A(D)\}$

$S \neq D$. $\Rightarrow \exists\, i_0$. s.t. $f_{i_0} \not\equiv 0$ on D.

WLOG. let $0 \in S$. $f_{i_0}(0', z_n) \not\equiv 0$.

$\Rightarrow \exists\ \{z' = 0.\ |z_n| = \delta_0\} \subset D/S$.

$\exists\ \{|z_j| < r_0,\ j \le n-1,\ |z_n| = \delta_0\} \subset D/S$.

Set: $U = \{|z_j| < r_0,\ j < n,\ |z_n| < \delta_0\}$.

Def: $F(z) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_0} \frac{f(z', \zeta) d\zeta}{\zeta - z_n}$ on U.

F is unif. $\Rightarrow F \in A(U)$.

3) Prove: $F|_{U/S} = f$

For $\forall$ fixed z'. $\{(z', z_n) \mid |z_n| \le \delta_0\} \cap S$ is finite discrete $\Rightarrow \varphi(z_n) \overset{\Delta}{=} f(z', z_n)$ is bdd and holomorphic except finite points. $\Rightarrow$ extend $\varphi(z_n)$ on $\{|z_n| \le \delta_0\}$.

So: $f(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_0} \frac{\varphi(\zeta) d\zeta}{\zeta - z_n} = F(z)$

Cor. $S \subsetneq D$ holomorphic $\Rightarrow$ D/S is connected

Pf: By contrad.: $D/S = U \cup V$.

Set $f(z) = \begin{cases} 1 : & U \\ 0 , & V \end{cases}$

can't be extended to D.

Cor. $f \in A(D)/\{0\}$. $Z(f.D) \neq \emptyset$. Then:

$Z(f,D)$ can't be compact.

Pf: $D/Z(f,D)$ is connected since Z holo.

Apply Hartogs as before.

Cor. $F \in A(D)$. If $\exists K \subset D$. cpt. D/K is domain and $F: D/K \xrightarrow{\sim} F(D/K)$ biholo.

Then: $F : D \xrightarrow{\sim} F(D)$ is also biholo.

Pf: $|F'(z)| \neq 0$ on D/K. $|F'(z)| \in A(D)$

$\Rightarrow$ $|F'(z)| \neq 0$ on D, by cor. above.

So: F is biholomorphic on D!

(4) Localization of Prin. Holo. Sets:

Def: For $A := A \subset A(D,S))$.

i) $Q := \{ f/g \mid f.g \in A\}$. quotient field of A.

Rmk: Note $fg \equiv 0 \Rightarrow f \equiv 0$ or $g \equiv 0$. Since

$f^{-1}(0) \cup g^{-1}(0)$ is nowhere dense. So:

There's no nilpotent in A. $\Rightarrow d$ is well-def.

ii) $A^0[w] := \{ w^n + \sum_0^{n-1} f_k w^k \mid f_k \in A \}$.

iii) $f \in \mathcal{O}[w]$ is irreducible if $f \neq f_1 f_2$. where f_1

$f_2 \in \mathcal{O}[w]$. $\deg f_1, f_2 > 1$.

Rmk: $\forall f \in \mathcal{O}[w]$. $f = \prod_1^n f_i^{k_i}$. f_i irreduci. in $\mathcal{O}[w]$.

Thm. If $u_1, u_2 \in \mathcal{O}^0[w]$. and $u_1 u_2 \in A^0[w]$. Then:

$u_1, u_2 \in A^0[w]$.

Pf: Lemma. For $w^n + a_1 w^{n-1} + \cdots a_n = 0$. $a_i \in \mathbb{C}^1$.

$\Rightarrow |w| \leq 2 \max_{1 \leq j \leq n} |a_j|^{1/j}$.

Pf: WLOG. set $a_n \neq 0$. otherwise ignore the roots $= 0$.

Set $w = \tilde{w}^{-1}$. Then:

$\sum_1^n a_k \tilde{w}^k + 1 = 0$ if $|\tilde{w}| \leq \cdots$

$\Rightarrow 1 = |\sum_1^n a_k \tilde{w}^k| < \sum 1/2^k < 1$.

Which is a contradiction!

$u = u_1 u_2 = \sum_1^{n-1} a_k w^k + w^n$. Since $a_k(z)$ are

bdd. By Lemma. $\Rightarrow$ root $(\varphi_k(z))$ of u

are also bdd. on $A(0, \delta)$

Set $u_k = \sum_{j=1}^{n_k} a_j^k w^j$. h is product of $(a_j^k)_{j,k}$'s denumerators.

Set $S = \{z \in \Delta(0,\delta) \mid h(z) = 0\}$.

$\Rightarrow a_j^k \in A(\Delta(0,\delta)/S)$ $\forall j, k$.

Besides, by vieta's and $Z(u) = Z(u_1) \cup Z(u_2)$

$\Rightarrow (a_j^k)$ can be expressed in $(y_\nu(z))$.

So $(a_j^k(z))$ are bdd. in nbd of S.

Apply Riemann's extension Thm. $\Rightarrow a_j^k \in A$.

<u>Cor.</u> $\forall u \in A^0[w]$. $u = \prod_1^N u_k$. $u_k \in A^0[w]$.

and it's unique.

<u>Pf:</u> Decompose u in $Q[w]$.

<u>Rmk:</u> We can extend def of irred. in $A^0[w]$.

<u>Def:</u> i) $f \in Q[w]$ has no multiple divisor if:

$f = \prod_1^N f_i$. $f_i \in Q[w]$. different up to unit in $Q[w]$.

ii) $Df := \sum_1^{n-1} i h_i w^{i-1} \in Q[w]$. for $f = \sum_0^N h_i w^i \in Q[w]$.

<u>Rmk:</u> If $f \in Q[w]$ has no multiple divisor. Then:

g.c.d $(f, Df) = h \not\equiv 0$ on $\Delta(0,\delta)$. i.e. unit in

④. Besides. $\exists\, q_1, q_2 \in A[w]$. s.t. $h = q_1 f + q_2 Df$

<u>Thm</u> $f \in \mathcal{O}^{n-1}[w_n]$ is Weierstrass poly w.r.t. w_n of degree $= k$. If f is irred. and $\exists$ $h \in \mathcal{O}^n$. s.t. $h = 0$ on $\{(w_1 \cdots w_n) \mid f = 0\}$

Then: $\exists\, q \in \mathcal{O}^n$. s.t. $h = qf$.

<u>Rmk</u>: It's alike the lemma of defining function w.r.t. manifold. We replace "$r(Df) = 1$" by "irred." here.

<u>pf</u>: $\exists\, \alpha, \beta \in \mathcal{O}^{n-1}[w_n]$. s.t. $\alpha f + \beta Df = r \not\equiv 0$.

So: $r(\tilde{z}) \neq 0 \Rightarrow f(\tilde{z})(z_n)$ has diff. roots.

By Weierstrass division: (w.r.t f)

$h = fq + h_0$. $h_0 \in \mathcal{O}^{n-1}[z_n]$. $\deg_{z_n} h < k$.

But $h(\tilde{z})(z_n) - f(\tilde{z})(z_n)\, q(\tilde{z})(z_n)$ has at least k diff roots of z_n. $\forall \tilde{z} \in \{r \neq 0\}$.

So: $h_0 \equiv 0$.

<u>Thm</u>. Denote: Resultant f, q is $\mathrm{Res}(f, q)$. and that $\Delta_f \overset{\Delta}{=} \mathrm{Res}(f, Df)$. For $w \in A^{\circ}[w]$. we have: w has no multiple divisor $\Leftrightarrow$ $\Delta_w \not\equiv 0$ on $A(0, \delta)$.

Pf: Recall: $\mathrm{Res}(f,g) = 0 \iff \deg(\text{g.c.d.}(f,g)) \geq 1$.

repeat the pf in Thm above.

Thm. (local Parametrization)

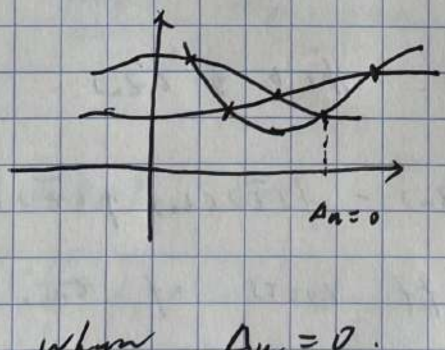
$u \in A^{\circ}[w]$. $\deg u = k$. has no multiple divisors in $A^{\circ}[w]$. $\pi: \mathbb{C}^n \times \mathbb{C}^1 \to \mathbb{C}^n$. canonical proj.

For $S := \{(z,w) \in \Delta(0,\delta) \times \mathbb{C}^1 \mid u(z,w) = 0\}$. and $Z_u := Z(\Delta_u, \Delta(0,\delta))$. we have:

i) $\forall z \in \Delta(0,\delta)$. $|S \cap \pi^{-1}(z)| = k$ if $z \in \Delta(0,\delta)/Z_u$.

$|S \cap \pi^{-1}(z)| < k$ if $z \in Z_u$.

ii) $S/\pi^{-1}(Z_u)$ is dense in S and is n-dim complex submanifold in $(\Delta(0,\delta)/Z_u) \times \mathbb{C}^1$.

Rmk: 

It's intuitive. the intersection points are the multiple roots.

when $\Delta_u = 0$. Discard these points. we get segments as manifolds.

Pf: i) $z_0 \notin Z_u \iff u(z_0, w)$ has k diff. roots.

ii) Note we can use nbd's to separate each zeros. and apply IFT.

Thm. Use the conditions of Thm. above. We have:

u is irred. in $A^0[W]$ $\Longleftrightarrow$ $S \cap (\Delta(0,\delta)/Z_u) \times \mathbb{C}'$

is connected complex submanifold.

Pf: ($\Rightarrow$) If $u = u_1 u_2$. $u_i \in A^0[W]$. We can

set $S_i \triangleq \{ u_i = 0 \}$. $S_1 \cap S_2 = \emptyset$.

$\Rightarrow$ $S \cap (\cdots) = S_1 \cup S_2$ isn't connected.

($\Leftarrow$) If $S \cap (\cdots) = \bigcup_1^N S_i$. connected compnt.

set: $S_i = \bigcup_1^{r_k} \{ (z, f_{ik}(z)) \}$.

$u_i = \prod_1^{r_k} (W - f_{ik}(z))$

Check: $u(z, W) = u_1(z,W) \cdots u_N(z,W)$.

Cor. Under the conditions above:

$u \in A^0[W]$. irred. $\Longleftrightarrow$ S_{reg} is connd manifold

Pf: Note $S \cap (\Delta(0,\delta)/Z_u) \subset S_{reg}$.

and dense in S_{reg}.