Holomorphic Convexity

(1) Reinhardt Domain:

Def: i) $\Omega \subset \mathbb{C}^n$ is Reinhardt domain with center 0 if $\forall z \in \Omega \Rightarrow \{(z_1 e^{i\theta_1}, \cdots z_n e^{i\theta_n}) \mid 0 \le \theta_i \le 2\pi\} \subset \Omega$.

ii) $\Omega \subset \mathbb{C}^n$ is complete Reinhardt (cen $= 0$) if $z \in \Omega \Rightarrow \{(z_1 \xi_1, \cdots z_n \xi_n) \mid |\xi_k| \le 1\} \subset \Omega$.

Rmk: i) is generalization of annulus and ii) is generalization of disc.

iii) D is Reinhardt domain with cen. $= 0$. if $\log D \triangleq \{(\log|z_1|, \cdots \log|z_n|) \mid z \in D / \{0\}\}$ is convex then we say D is logrithm convex.

Rmk: i) Set $\widetilde{D} = \{ r_1^t r_2^{1-t} \mid r_1, r_2 \in D, t \le 1\}$.

$\Rightarrow \widetilde{D}$ is min log-convex Reinhardt containing D.

ii) log-convex $\not\Rightarrow$ Complete. e.g. $\{1 < |z| < 2\}$.

complete $\not\Rightarrow$ log-convex. e.g. $\{|z_1| < 1, |z_2| < 2\} \cup \{|z_2| < 2, |z_1| < 1\} \subset \mathbb{C}^2$.

Thm. $D \underset{\text{dom.}}{\subset} \mathbb{C}^n$. complete Reinhardt, with center $= 0$.

$f \in A(D) \Rightarrow P(0, f) = \sum_\alpha \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha$ converges

on $\forall z \in \tilde{D}$. (Ref in Rmk above.)

Pf: $\forall \zeta \in \tilde{D}$. $r \triangleq (|\zeta_1|, \cdots |\zeta_n|) = (r_1^{1-t} r_1'^{t}, \cdots)$

$\Rightarrow \left|\frac{f^{(\alpha)}(0)}{\alpha!}\right| r^\alpha = \left(\left|\frac{f^{(\alpha)}(0)}{\alpha!}\right| r^{\alpha}\right)^{1-t} \left(\left|\frac{f^{(\alpha)}(0)}{\alpha!}\right| r'^{\alpha}\right)^t$

$\leq M < \infty$.

So: $P(0, f)$ converges in $\Delta(0, r)$.

$\Rightarrow P(0, f)$ converges in $\tilde{D} \subset \bigcup_{\zeta \in \tilde{D}} \Delta(0, |\zeta|)$.

Lemma. $D \underset{\text{dom.}}{\subset} \mathbb{C}^n$. complete Reinhardt. logarithm convex

with center $= 0$. $E \subset\subset D$. $a \in \mathbb{C}^n / D$. Then.

$\exists\, m_a(z)$ monomial. st. $\sup_E |m_a| \leq m_a(a) = 1$

Pf: 1°) $E \subset\subset D = \bigcup_{\zeta \in D} \Delta(0, (|\zeta_1|, \cdots |\zeta_n|))$.

So $\exists\, (\zeta^{(k)})_1^m$. st. $E \subset \bigcup_1^m \Delta(0, (|\zeta_1^{(k)}| \cdots |\zeta_n^{(k)}|))$

Besides. $\sup_E |m(z)| \leq \max_{1 \leq k \leq m} |m(\zeta^{(k)})|$. $\forall m$. monomial.

2) By convexity of $\log D$. $\exists\, l(x) = r \cdot x + \delta$.

st. $\begin{cases} \sum_1^n r_i x_i + \delta < 0. & \forall x \in \log D \subset \mathbb{R}^n \\ \sum_1^n r_i |a_i| + \delta = 0. \end{cases}$

$\Rightarrow r_i \geq 0$. $\forall i \leq n$. since $\xi \in \log D \Rightarrow \{\tilde{\xi} \in \mathbb{R}^n \mid$

$\tilde{\xi}_i \leq \xi_i,\ \forall i\} \subset \log D$. (Otherwise, set $x_k \to -\infty$)

By density. $\exists\, l_i \in \mathbb{Z}^+$. $\delta_0 \in \mathbb{R}'$. s.t.

$$\begin{cases} \max_k \left(\sum_i^n l_i \xi_i^{(k)} + \delta_0 \right) < 0 \\ e^{\delta_0} \cdot |a^l| = 1. \end{cases}$$

Set $m_a(z) = a^{-l} z^l$. by i). it holds!

Thm. Under the conditions above. Set M is set of holo. monomials. $\Rightarrow \{ z \in D \mid |m(z)| \le \sup_E |m|.\ \forall m \in M \}$ $=: D^* \subset\subset D$.

Pf: Note $E \subset\subset D$. Set $m = z_k$. $\Rightarrow D^*$ is bdd closed.

D^* isn't cpt set of $D \iff \exists\, (p_k) \subset D^*$. s.t. $p_k \to \partial D$. WLOG. (p_k) is subseq $\to a \in \partial D$.

So: $|m(p_j)| \to |m(a)| \le \sup_E |m|$. $\forall m \in M$.

Contradict with Lemma above!

(2) Holomorphic Convex Domain:

Def: i) In topo space X. $F(X)$ is some set of real func. If $K \subset\subset X$. $\hat{K}_{F(X)} := \{ x \in X \mid f(x) \le \sup_K f.$ for $\forall f \in F(X) \} \subset\subset X$. Then we say X is convex w.r.t $F(X)$ in X. ($F(X)$ - convex.).

Rmk: By Thm. above. D is logrithm convex. complete Reinhardt domain with cen

$= 0 \Rightarrow D$ is M-convex in $\mathbb{C}^n$.

ii) Set $\hat{K}_\Omega \triangleq \hat{K}_{A(\Omega)} = \{ z \in \Omega \mid |f(z)| \leq \sup_K |f|$, for $\forall f \in A(\Omega) \}$. for $\Omega \subset \mathbb{C}^n$ domain. If $\hat{K}_\Omega \subset\subset \Omega$ for $\forall K \subset\subset \Omega$. Then we say Ω is holomorphic convex.

<u>Thm.</u> When $X = \mathbb{R}^n$. $\mathcal{L}(X)$ is set of linear function on $\mathbb{R}^n$. Then $\mathcal{L}(X)$-convex set is equi. with (Euclidean) convex set.

Next, we fix $\Omega \subset \mathbb{C}^n$ domain. Denote $\hat{K} \triangleq \hat{K}_\Omega$.

<u>Prop.</u> For $K \subset \Omega$. we have:

i) $K \subset \hat{K}$ ii) $\hat{K}$ is cpt in Ω. iii) $\hat{\hat{K}} = \hat{K}$.

iv) $K_1 \subset K_2 \subset \Omega \Rightarrow \hat{K}_1 \subset \hat{K}_2 \subset \Omega$.

v) K is bdd $\Rightarrow \hat{K}$ is bdd.

<u>Pf:</u> ii) By conti. of $f \in A(\Omega)$. $\Omega / \hat{K}$ is open.

iii) Note: $\sup_{\hat{K}} |f(z)| = \sup_K |f(z)|$.

v) $\exists R > 0$. $K \subset \{ |z_i| \leq R, \forall i \leq n \}$.

Set $f = z_k \in A(\Omega)$. $\xRightarrow[\in \hat{K}]{\forall z} |z_k| \leq R$.

<u>Lemma</u> If Ω is holomorphic convex. Then $\exists (K_j)$. cpt seq. in Ω. s.t. i) $K_j \subset \operatorname{int} K_{j+1}$ ii) $\Omega = \bigcup K_j$

iii) $\hat{K}_j = K_j$.

Pf: $\exists (\tilde{K}_j)$. cpt seq. st. $\tilde{K}_j \subset \mathring{\tilde{K}}_{j+1}$.

and $\Omega = \cup \tilde{K}_j$.

Choose seq. (K_j) from $(\hat{\tilde{K}}_j)$.

st. $K_j = \hat{\tilde{K}}_{i_j} \subset \mathring{\tilde{K}}_{i_{j+1}} = \mathring{K}_{j+1}$.

Besides. holo. convex. $\Rightarrow (K_j)$ is cpt.

Lemma^2. $K \subset \Omega$. $\forall \varepsilon > 0$. $M > 0$. $p \in \Omega / \hat{K}$. $\exists f \in A(\Omega)$.

St. $\sup_K |f| < \varepsilon$. $|f(p)| > M$.

Pf: $\exists h \in A(\Omega)$. St. $\sup_K |h| < |h(p)|$.

Choose δ_0. st. $\sup_K |h| < \delta_0 < |h(p)|$.

Set $g(z) = h(z)/\delta_0$. $\exists m \in \mathbb{Z}^+$. large

St. $f = g^m$ satisfies the condition.

Lemma^3. For $(K_j) \subset \Omega$. is cpt seq in Lem^1.

If $p_j \in K_{j+1}/K_j$. Then $\exists f \in A(\Omega)$. St.

$\lim_j |f(p_j)| = \infty$.

Pf: By Lem^2. $\exists f_j \in A(\Omega)$. St.

$$\begin{cases} \sup_{K_j} |f_j| < 1/2^j \\ |f_j(p_j)| > j+1 + \sum_1^{j-1} |f_i(p_j)|. \end{cases}$$

Set $f = \sum_1^\infty f_j$ satisfies the cond.

Thm. G is holo. convex. $\iff$ $\forall (p_j) \to \partial G$. $(p_j) \subset G$.

$\exists f \in A(G)$. s.t. $(f(p_j))$ is unbdd.

Pf: $(\Rightarrow)$. By Lemma 3.

$(\Leftarrow)$. If $\hat{K}$ isn't cpt in G. Then:

$\exists (p_j) \subset \hat{K} \to \partial G$. $|f(p_j)| \overset{\forall j}{\le} \sup_K |f| < \infty$.

Cor. i) Connected domain is holo. convex. in $\mathbb{C}^1$.

ii) Euclidean convex. domain in $\mathbb{C}^n$

is holo. convex.

Pf: i) Set $1/(z-z_0) = f$. $z_0 \in \partial G$.

ii) $\exists L(z)$. linear function. st.

$\{L(z)=0\} \cap \partial G \ni p$. for $p \in \partial G$.

and $\{L(z)=0\} \cap G = \emptyset$.

Set $f(z) = 1/L(z)$.

(3) Domain of Holomorphy:

Def: $\Omega \subset \mathbb{C}^n$. domain.

i) Ω is domain of holomorphy if there doesn't exist domains Ω_1. Ω_2. st. $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$.

$\Omega_2 / \Omega \neq \emptyset$. and $\forall f \in A(\Omega)$. $\exists h \in A(\Omega_2)$. st.

$f|_{\Omega_1} = h|_{\Omega_1}$.

Rmk: It means $\exists f$ can't extend holo outside Ω.

ii) $p \in \partial\Omega$ is singular w.r.t. $f \in A(\Omega)$ if $\forall$ $U(p)$ nbd of p. f can't extend holo. on $U(p)$. If $\forall p \in \partial\Omega$ is singular w.r.t f. we say $\partial\Omega$ is natural boundary.

Rmk: If $\forall p \in \partial\Omega$. $\exists f \in A(\Omega)$. st. p is singular w.r.t f. $\Rightarrow$ Ω is domain of holomorphy.

Lemma. $G \subset \mathbb{C}^n$ domain. If (K_j) cpt seq of G satisfies: $G = \bigcup K_j$. $K_j \subset \operatorname{int} K_{j+1}$. Then: $\exists$ $(p_k) \subset G$. st. $p_k \in K_{j_{k+1}} / K_{j_k}$. $\forall k$ and. $\forall p \in \partial G$. $\forall U(p)$ nbd of p. $\forall$ connected comp. of $U(p) \cap G$ contains infinite points of (p_k).

Pf. $B = \{ B(a,r) \cap \partial G \neq \emptyset \mid a, r \in \mathbb{Q}^n \}$.

$A = \{ D_k : \exists B(a,r) \in B.$ st. D_k is connected comp. of $G \cap B(a,r) \}$.

Note $B(a,r) \cap \partial G \cap \partial D_k \neq \emptyset$. So:

$\forall$ cpt $K \subset G$. $D_k / K \neq \emptyset$.

take $p_1 \in D_1 \subset \bigcup (K_{j+1} / K_j)$.

($\exists\, j_1$ s.t. $p_1 \in K_{j_1+1} / K_{j_1}$).

take $p_2 \in D_2 / K_{j_1+1}$ .... (p_k) satisfies it!

<u>Cor.</u> $G \subset \mathbb{C}^n$ is holomorphic convex domain. Then $\exists f \in A(G)$. s.t. ∂G is natural boundary of f.

<u>Pf</u>: With Lem$^?$ in (2).

<u>Cor.</u> Holomorphic convex domain is domain of holomorphy.

<u>Lemma</u> $G \subset \mathbb{C}^n$ is domain of holomorphy. $K \subset\subset G$.

Then: $d(K, \partial G) = d(\hat{K}, \partial G)$.

<u>Pf.</u> Note $d(K, \partial G) > \varepsilon > 0$. Set $K_\varepsilon = K + \varepsilon$.

$\forall f \in A(G)$. By Cauchy formula. ($\Delta(z, \varepsilon)$).

$|f^{(\alpha)}(z)| \leq \frac{\alpha!}{\varepsilon^{|\alpha|}} \sup_{\overline{K_\varepsilon}} |f|$. $\forall z \in K$.

$\Rightarrow \forall a \in \hat{K}$. $|f^{(\alpha)}(a)| \leq \frac{\alpha!}{\varepsilon^{|\alpha|}} \sup_{\overline{K_\varepsilon}} |f|$.

So: $\sum \frac{f^{(\alpha)}(a) (z-a)^\alpha}{\alpha!} = f(z)$ converges in

$A(a, \varepsilon) \Rightarrow d(a, \partial G) \geq \varepsilon$. i.e. $d(\hat{K}, \partial G) \geq \varepsilon$

<u>Cor.</u> Domain of holo. is holo. convex.

Thm. (Cartan - Thullen)

$G \subset \mathbb{C}^n$. The follows equi. :

i) G is domain of holomorphy

ii) $\forall K \subset\subset G$. $d(K, \partial G) = d(\hat{K}, \partial G)$.

iii) G is holo. convex. iv) ∂G is natural boundary.

Thm. $D \subset \mathbb{C}^n$ is complete Reinhardt domain

with cen. $= 0$. The follows equi. :

i) D is converge domain of some power

series : $\sum C_r z^r$.

ii) D is logarithm convex.

iii) D is M- convex. iv) D is holo. convex.

Rmk: Most holomorphic func's on G have ∂G

as their natural boundary.

Denote the set of such func. by N. Then:

$A(G)/N$ is nowhere dense in $A(G)$.