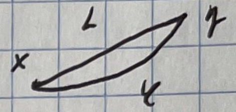
Plurisubharmonicity.

(1) Subharmonicity on Plane:

Def: i) $\varphi : (a,b) \to \mathbb{R}'$ is convex if $\forall \lambda \in [0,1]$. $x, y \in (a,b)$. We have $\varphi((1-\lambda)x + \lambda y) \le (1-\lambda)\varphi(x) + \lambda \varphi(y)$.



Rmk: It's equi.:

$\forall [x,y] \subset\subset (a,b)$. $\forall L(z)$ linear.

$L|_{\partial[x,y]} \ge \varphi|_{\partial[x,y]} \Rightarrow L|_{[x,y]} \ge \varphi|_{[x,y]}$.

Cor. φ has upper bdd and convex.

$\Rightarrow \varphi \equiv$ const.

Pf: If $\varphi \not\equiv$ const. $\Rightarrow \varphi' > 0$. $\uparrow$ a.e.

ii) For $D \subset \mathbb{C}$. $u : D \to [-\infty, +\infty)$. u.s.c.

u is subharmonic on D if $\forall \Delta(z,r) \subset\subset D$.

$\forall h \in C(\bar{\Delta}(z,r))$ harmonic. $h|_{\partial\Delta} \ge u|_{\partial\Delta}$

$\Rightarrow h|_{\Delta(z,r)} \ge u|_{\Delta(z,r)}$.

Prop. $u : D \to [-\infty, +\infty)$. u.s.c. Then:

u is subharmonic $\iff \forall a \in D$. $\exists$ nbd $U_a \subset D$

s.t. $u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta$. $\forall \bar{\Delta}(a,r) \subset U_a$.

Pf: ($\Rightarrow$). $\exists (u_n) \subset C(\partial D) \searrow u|_{\partial D}$.

extend u_n on D st. harmonic.

By MCT: $\frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta}) = \lim_n \int \square$

$= \lim_n u_n(a) \geq u(a)$.

($\Leftarrow$). $\forall \varphi$ harmonic on $\overline{\Delta(z,r)}$. $\varphi|_{\partial D} \geq u|_{\partial D}$

apply the formula on $u-\varphi$.

Prop. φ is subharmonic on domain D. For $\overline{\Delta}(z_0,r)$

$\subset D$. Set $M(\varphi,t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0+te^{i\theta})d\theta$.

$\Rightarrow M(\varphi,t) \nearrow$ on $t \in (0,r)$.

Pf: $\forall 0<r_1<r_2<r$. $\varphi|_{\partial\Delta(z_0,r_2)} \leq h|_{\partial\Delta(z_0,r_2)}$.

$h \in C(\overline{\Delta}(z_0,r_2))$ harmonic.

$\Rightarrow M(\varphi,r_1) \leq M(h,r_1) = M(h,r_2)$.

So: $M(\varphi,r_1) \leq \inf\{M(h,r_2) \mid h \in C(\partial\Delta(z_0,r_2))$.

$h|_{\partial\Delta(r_2)} \geq \varphi|_{\partial\Delta(r_2)}\}$.

$= M(\varphi,r_2)$.

prop. $\varphi: D \to \mathbb{R}$ on $D \underset{\text{dom.}}{\subset} \mathbb{C}$. φ is harmonic

$\Leftrightarrow -\varphi, \varphi$ are subharmonic.

Pf: Consider $\varphi|_{\partial\Delta} = h|_{\partial\Delta}$. h harmonic. By def.

prop. $D \subset \mathbb{C}^1$. domain.

i) φ_1, φ_2 subharmonic on D. $c > 0 \Rightarrow \max_{1,2} \varphi_i$

$c\varphi_1$. $\varphi_1 + \varphi_2$ are subharmonic.

ii) $(u_\alpha)_{\alpha \in A}$ is family of subharmonic. If $u = \sup_\alpha u_\alpha$ is finite. u.s.c. then u is subharmonic.

iii) $(u_j) \downarrow$ subharmonic $\Rightarrow \lim_j u_j$ is subharmonic.

Thm. $D \underset{\text{dom.}}{\subset} \mathbb{C}^1$. $u \in C^2(D, \mathbb{R}^1)$. u is subharmonic $\Leftrightarrow \Delta u \geq 0$.

Pf: $(\Leftarrow)$. WLOG. Set $\Delta u > 0$. otherwise.

Set $u_j = u + \frac{1}{j}|z|^2$. $\Delta u_j > 0$. $u_j \downarrow u$.

For $\bar{\Delta}(a,r) \subset\subset D$. $h \in C(\bar{\Delta}(a,r))$.

harmonic. $v = u - h \leq 0$ on $\partial\Delta(a,r)$.

If $\exists z_0 \in \Delta(a,r)$. $v(z_0) > 0 \geq v|_{\partial\Delta}$.

$\Rightarrow \exists p \in \Delta(a,r)$. s.t. $v(p)$ attain max.

$\Rightarrow \frac{\partial^2 v}{\partial x^2}(p)$, $\frac{\partial^2 v}{\partial y^2}(p) \leq 0$. $\Rightarrow \Delta u(p) \leq 0$.

$(\Rightarrow)$. If $\exists a \in D$. $\Delta u(a) < 0$. $\Rightarrow \exists U_a$. nbd.

$\Delta u < 0$ on U_a. $\Rightarrow -u$ subharmonic

$\Rightarrow u$ harmonic. $\Rightarrow \Delta u \equiv 0$. contradict!

prop. φ defined on $\mathbb{R}^1$. $\uparrow$. convex. u is subharmonic

on $D \underset{\text{dom}}{\subset} \mathbb{C}^1 \Rightarrow \varphi \circ u$ is subharmonic on D.

Pf: $\varphi\circ u$ is u.s.c is easy to see.

With $\varphi(u(a+re^{i\theta})) \geq \varphi(u(a)) + k(u(a+re^{i\theta}) - u(a))$

$\Rightarrow$ we have submean value inequi.

(2) Convex on $\mathbb{R}^p$:

Def: For $\Omega \subset \mathbb{R}^p$. convex open domain. $f: \Omega \to \mathbb{R}'$ is convex if $\forall p_1, p_2 \in \Omega$. $\lambda \in [0,1]$.

$f(\lambda p_1 + (1-\lambda) p_2) \leq \lambda f(p_1) + (1-\lambda) f(p_2)$.

Lemma f is convex on $\Omega \Leftrightarrow \forall x \in \Omega$. $\xi \in \mathbb{R}^p$.

$$\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot \xi_i \xi_j \geq 0.$$

Pf: Note $\frac{d^2 f(x+t\xi)}{dt^2} \Big|_{t=0} \geq 0$.

prop. $(\varphi_i)_1^p$ are subharmonic. If $h: \mathbb{R}^p \to \mathbb{R}'$. is convex. $\uparrow$. can be extended to $[-\infty, +\infty)^p \xrightarrow{\text{conti.}} [-\infty, +\infty)$. Then: $h(\varphi_1, \cdots \varphi_p)$ is also subharmonic on $D \subset \mathbb{C}'$.

Pf: 1) u.s.c. is from the conti. extension.

2) Note $h(t_1 \cdots t_p) = \sup_{\alpha \in A} \alpha(t_1, \cdots t_p)$.

α is linear, st. $\alpha \leq h$.

Cor. $(\varphi_i)_1^p$ is subharmonic on $D \subset \mathbb{C}^1$.

Then: $\log(e^{\varphi_1} + \cdots + e^{\varphi_p})$ is subharmonic.

Cor. i) $f \in A(D) \Rightarrow \log|f|$ is subharmonic.

ii) $(f_i)_1^p \subset A(D)$. $(r_i) \subset \mathbb{R}^{>0}$. Then:

$u(z) = \log(\sum_1^p |f_i|^{r_i})$ is subharmonic.

Pf: i) $\frac{\partial^2}{\partial z \partial \bar{z}} \log|f| = 0$ for $z \in \{f \neq 0\}$.

ii) $u(z) = \log(\sum_1^p e^{r_i \log|f_i|})$

(3) Plurisubharmonic:

Def: i) $\Omega \underset{\text{dom.}}{\subset} \mathbb{C}^n$. $\varphi : \Omega \to [-\infty, +\infty)$. u.s.c is subharmonic if $\forall x_0 \in \Omega$. $\exists$ nbd $U(x_0) \subset \Omega$.

s.t. $\varphi(x_0) \le \fint_{\partial B(x_0, r)} \varphi \, ds$. $\forall \bar{B}(x_0, r) \subset U(x_0)$.

Denote the set of such func. by $SH(\Omega)$.

ii) $D \underset{\text{dom}}{\subset} \mathbb{C}^n$. $u : D \to [-\infty, +\infty)$. u.s.c. is plurisubharmonic if $\forall a \in D$. $w \in \mathbb{C}^n$. $\lambda \mapsto u(a + \lambda w)$ is subharmonic on $\{\lambda \in \mathbb{C}^1 \mid a + \lambda w \in D\}$. Denote the set by $PSH(D)$.

Rmk: $PSH(D) \subset SH(D)$ if D is domain.

(Note: $u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta} \cdot w) \, d\theta$

prop. $\Omega \underset{\text{dom}}{\subset} \mathbb{C}^n$.

i) $\varphi, \psi \in PSH(\Omega)$, $c > 0 \Rightarrow \max\{\varphi, \psi\}$, $c\varphi$, $\varphi + \psi \in PSH(\Omega)$

ii) $(\varphi_\lambda)_{\lambda \in \Lambda}$ is family in $PSH(\Omega)$. If $\varphi = \sup_\lambda \varphi_\lambda$ is finite, u.s.c. $\Rightarrow \varphi \in PSH(\Omega)$.

iii) $(\varphi_j) \subset PSH(\Omega)$, $\downarrow$. $\Rightarrow \lim_j \varphi_j \in PSH(\Omega)$.

iv) $\varphi \in C^2(\Omega, \mathbb{R}^1)$. Then $\varphi \in PSH(\Omega) \iff$ $\left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_k}\right)$ nonnegative definite Hermite.

v) $\varphi \in PSH(\Omega)$, $\varphi(\Omega) \subset (a, b)$. $h: (a, b) \to \mathbb{R}^1$, $\uparrow$, convex $\Rightarrow h \circ \varphi \in PSH(\Omega)$.

Pf: From properties of subharmonic func.

prop. $\Omega \subset \mathbb{C}^n$ domain. $u \in PSH(\Omega) / \{-\infty\} \Rightarrow u \in L^1_{loc}(\Omega)$.

Pf: $\forall \bar{\Delta}(a, r) \subset \Omega$. Note u has upper bdd on $\Delta(a, r)$. So: $u \in L^1(\Delta) \iff \int_{\Delta(a,r)} u > -\infty$.

$$LHS = \int_0^{r_1} e_1 de_1 \cdots \int_0^{r_n} e_n de_n \int_0^{2\pi} \cdots \int_0^{2\pi} u(a_1 + e_1 e^{i\theta_1}, \cdots a_n + e_n e^{i\theta_n}) d\theta_1 \cdots d\theta_n$$

$$\overset{\text{subhar.}}{\geq} (2\pi)^n \int_0^{r_1} e_1 de_1 \cdots \int_0^{r_n} e_n de_n \cdot u(a)$$

$$= c \cdot u(a).$$

Set $E = \{ z \in \Omega \mid u(z) \text{ is locally } L^1\text{-integrable} \}$.

$\Rightarrow$ E is open nonempty set.

By inequi. above. Ω / E is open.

$\Rightarrow$ E is clopen in Ω. So: $E = \Omega$.

<u>prop.</u> $\Omega \underset{\text{dom}}{\subset} \mathbb{C}^n$. $\varphi \in PSH(\Omega) / \{-\infty\}$. Then $\varphi_\varepsilon(z) = \int_{\mathbb{C}^n} \varphi(z+y)\, \rho_\varepsilon(y)\, dy \in PSH(\Omega_\varepsilon)$. where (ρ_ε) is mollifier. Besides. $\forall z \in \Omega$. $\varphi_\varepsilon(z) \underset{\varepsilon \to 0}{\downarrow} \varphi(z)$.

<u>Pf:</u> 1) Check (φ_ε) satisfies submean value.

2) Check $\varphi_{\varepsilon_1} \geq \varphi_{\varepsilon_2}$. $\forall \varepsilon_2 \leq \varepsilon_1$. by 2^{nd} prop in (1)

3) With Fatou, prove: $\varphi_\varepsilon \downarrow \varphi$.

<u>Cor.</u> $\varphi, \psi \in PSH(\Omega) / \{-\infty\}$. $\varphi = \psi$. a.e. $\Rightarrow \varphi = \psi$.

<u>Pf:</u> Note $\varphi_\varepsilon = \psi_\varepsilon$. $\downarrow$ $\varphi = \psi$. pointwise.

<u>Cor.</u> $\Omega_1 \subset \mathbb{C}^n$. $\Omega_2 \subset \mathbb{C}^m$. domain. If $F : \Omega_1 \to \Omega_2$ holomorphic. $\varphi \in PSH(\Omega_2) \Rightarrow \varphi \circ F \in PSH(\Omega_1)$.

<u>Pf:</u> Check $\sum_{i,j} \dfrac{\partial^2 \varphi_\varepsilon(F(z))}{\partial z_i\, \partial \bar{z}_j} \xi_i \bar{\xi}_j \geq 0$.

So: $\varphi_\varepsilon \circ F \downarrow \varphi \circ F \in PSH(\Omega_1)$

<u>Def:</u> $D \subset \mathbb{C}^n$. domain. $u : D \to \mathbb{R}^1$ is pluriharmonic (PH). if $u, -u \in PSH(D)$.

Rmk: $u \in PH(D) \Rightarrow u$ is conti. $\Rightarrow$

By Poisson formula $u \in C^\infty$.

Thm. $\varphi \in C^2(D)$. Then: $\varphi \in PH(D) \Leftrightarrow \partial\bar{\partial}\varphi \equiv 0$

Pf: $\partial\bar{\partial}\varphi,\ \partial\bar{\partial}(-\varphi) \geq 0 \Leftrightarrow \partial\bar{\partial}\varphi \equiv 0$.

Rmk: $f \in A(D) \Rightarrow \partial\bar{\partial}f = \partial\bar{\partial}\bar{f} = 0$.

So: $\mathrm{Re}f,\ \mathrm{Im}f \in PH(D)$.

Thm. $D \subset \mathbb{C}^n$ domain satisfies $H^1_{DR}(D) = 0$

Then: $u \in PH(D) \Rightarrow \exists f \in A(D)$. s.t. $\mathrm{Re}f = u$.

Pf: $d(\partial u) = \bar{\partial}\partial u = 0$. So $\exists f \in C^\infty(D)$.

s.t. $df = \partial u \in \Lambda^{1,0}(D)$. $\Rightarrow \bar{\partial}f = 0$.

With $d(f+\bar{f}-u) = (df - \partial u) + (d\bar{f} - \bar{\partial}u) = 0$.

Def: $\Omega \underset{\text{dom}}{\subset} \mathbb{C}^n$. $A \subset \Omega$ is multipole set if $\forall z \in \Omega$.

$\exists W_z$. nbd $\subset \Omega$. $u \in PSH(W_z)/\{-\infty\}$. s.t. $A \cap W_z$

$\subset \{z' \in W_z \mid u(z') = -\infty\}$.

Rmk: Any holo. set is multipole. (So null measure).

Thm. $A \subset \Omega \underset{\text{dom}}{\subset} \mathbb{C}^n$. closed multipole set. If $v \in PSH(\Omega/A)$.

$\forall x_0 \in A$. $\exists$ nbd $W_{x_0} \subset \Omega$. s.t. $v|_{W(x_0)/A}$ has upper bdd.

Then $\exists$ unique $\tilde{v} \in PSH(\Omega)$. s.t. $\tilde{v}|_{\Omega/A} = v$.

Cor. Replace "$PSH(\Omega/A)$. $PSH(\Omega)$" by "$A(\Omega/A)$. $A(\Omega)$".