# Preliminary

(1) Intro.:

Mean field is high symmetry form of interaction between particles which are represented by SDEs.

Topics: Move from micro. to macro.

① Interact. Diffusion:

$(X_t^i)$ are Itô processes. defined by

$$: dX_t^i = b(X_t^i, \hat{m}_t)dt + \sigma(X_t^i, \hat{m}_t)dW_t^i$$

$$\hat{m}_t = \sum_1^n \delta_{X_t^k} / n.$$

where $(W^i)$ are i.i.d. BMs.

Rmk: The system is symmetric so $(X^i)_1^n$ are exchangable.

$\Rightarrow$ goal: Analyse the $n$-limit. $X^\infty$

② Mean field game:

Next, we consider the controlled version:

$$dX_t^i = b(X_t^i, \hat{m}_t, \alpha_t^i)dt + \sigma(X_t^i, \hat{m}_t, \alpha_t^i)dW_t^i$$

where $(\alpha^i)$ are chosen control process.

Def: We call $(X_t^i)$ by agents.

And we choose $(X_t^i)$ to maximizes:

$$J_i^n(\alpha^1 \cdots \alpha^n) = \mathbb{E}\Big(\int_0^T f(X_t^i, \hat{m}_t, \alpha_t^i)dt + g(X_T^i, \hat{m}_T)\Big).$$

$\Rightarrow$ Study the $n$-limit of $X_t^n$.

(2) Converge and Metric:

Equip metric space $(X, d)$ with Borel $\sigma$-algebra.

Denote: $\mathcal{P}(X)$ is set of p.m.'s on $X$.

① Thm. (Prokhorov's)

$(\mu_n)$ is tight in $\mathcal{P}(X)$. $\underset{X \text{ is polish}}{\rightleftarrows}$ $(\mu_n)$ is precpt in $\mathcal{P}(X)$.

Thm (Skorokhod's)

If $(X, d)$ is separable. $\mu_n \xrightarrow{w} \mu$.

Then: $\exists (\Omega, \mathcal{F}, \mathbb{P})$ supporting $X$-valued r.v.'s $(X_n) \sim \mu_n$. and $X \sim \mu$, st. $X_n \to X$. a.s.

<u>Thm.</u> (For empirical measure)

$(X_k) \overset{i.i.d}{\sim} \mu$. If $(X, d)$ is separable.

Then: $\mu_n := \sum_1^n \delta_{X_k} / n \to \mu$.

<u>Pf:</u> Check on dense countable family $(f_n) \subset C_b(X)$.

Apply SLLN on $\int_X f_k \, d\mu_n$

② Next, we assume $(X, d)$ is separable.

<u>Def:</u> i) $P^p(X) := \{ \mu \in P(X) \mid \int_X d(x_0, x)^p d\mu < \infty \}$. $x_0$ is fixed. $p \geq 1$.

<u>Rmk:</u> It's indept. of choice of $x_0$.

ii) $W_p(\mu, \nu) := (\inf_{\substack{X \sim \mu \\ Y \sim \nu}} E(d(X, Y)^p))^{\frac{1}{p}}$

p-Wass. distance on $P^p(X)$.

Rmk: Since $W_p$ involves infimum. $\Rightarrow$ it's easy to bound.

Thm. If $(X,d)$ is complete and separable Then: $W_p$ is metric on $P^p(X)$ and $(P^p(X), W_p)$ is complete and separable

Rmk: $W_p \leq W_q$. if $p \leq q$. So: $P^p \supset P^q$.

Thm. (Kantorovich duality)

If $(X,d)$ is polish. Then. for $p \geq 1$.

$$W_p^p(\mu,\nu) = \sup \{ \int_X f\, d\mu + \int_X g\, d\nu \mid f,g \in C_B(X).\ f(x)+g(y) \leq d(x,y)^p\ \forall x,y \}.$$

Cor. For $p=1$. we have:

$$W_1(\mu,\nu) = \sup \{ \int f\, d\mu - \int f\, d\nu \mid f \in Lip^1(X) \}.$$

Thm. C Characterization of $W_p$-converg:

If $(\mu_n), \mu \in \mathcal{P}^p(X)$, $p \geq 1$. Then:

i) $W_p(\mu_n, \mu) \to 0$

ii) $\mu_n \xrightarrow{w} \mu$. $[\int_X d(x, x_0)^p d\mu_n \to \int_X d(x, x_0)^p d\mu]$ or $[u.i.]$.

iii) $\forall f \in C(X, \mathbb{R})$. s.t. $\exists x_0$. $c > 0$. s.t. $|f(x)| \leq c(1 + d(x, x_0)^p)$. $\forall x \in X$.

$\Rightarrow \int_X f \, d\mu_n \to \int_X f \, d\mu$.

We have i). ii). iii) equi.

Rmk: If replace $d$ by $\bar{d} := 1 \wedge d$.

$\Rightarrow \overline{W}_p$ is bdd metric $\Rightarrow$ ii) ✓.

So: $\mu_n \xrightarrow{w} \mu \iff \overline{W}_p(\mu_n, \mu) \to 0$.

Cor. $(X_k) \overset{i.i.d}{\sim} \mu$. $\in \mathcal{P}^p(X)$. $X$-valued r.v.'s.

For $\mu_n = \sum_1^n \delta_{X_k} / n$. we have:

i) $W_p(\mu_n, \mu) \to 0$. a.s.

ii) $\mathbb{E}(W_p(\mu_n, \mu)) \to 0$.

Pf: Lemma. (Poussin)

For $X \in L^1$. $\mathbb{R}^+$-valued. Then:

$\exists\, \varphi: \mathbb{R}^+ \to \mathbb{R}^+$. $\uparrow$. convex. st.

$\lim\limits_{t\to\infty} \varphi(t)/t = \infty$. and $\mathbb{E}(\varphi(X)) < \infty$.

Cor. $(Z_i) \in L^1$ i.i.d. $\mathbb{R}^1$-valued

$\Rightarrow S_n = \frac{1}{n}\sum_1^n Z_k$ is u.i.

Pf: WLOG. set $Z_i \geq 0$.

By Lemma. :

$$\sup_n \mathbb{E}(\varphi(S_n)) \overset{\text{convex}}{\leq} \mathbb{E}(\varphi(X_1))$$

i) $\lim\limits_{n\to\infty} \int d(x,x_0)^p \, d\mu_n = \lim\limits_n \frac{1}{n}\sum_1^n d(X_k, X_0)^p$

$\overset{\text{SLLN}}{=} \mathbb{E}(d(X_1, X_0)^p)$.

With $\mu_n \xrightarrow{w} \mu$. $\Rightarrow W_p(\mu_n, \mu) \xrightarrow{a.s.} 0$

ii) By $C_p$-inequality :

$W_p^p(\mu_n, \mu) \leq 2^{p-1}(W_p^p(\mu_n, \delta_{x_0}) + W_p^p(\delta_{x_0}, \mu))$

$= \frac{2^{p-1}}{n}\sum_1^n d(X_i, X_0)^p + \square \overset{\text{Lem.}}{\Rightarrow}$ u.i.

(3) Stochastic Control:

Def: $dX_s^{t,x,\alpha} = b(X_s^{t,x,\alpha}, \alpha(s))ds + \sigma(X_s^{t,x,\alpha}, \alpha(s))dW_s$

where $b, \sigma$ satisfy E & U cond's.

Remark: i) $\mathbb{A} := \{ (\alpha_t) \text{ progressive} \mid \mathbb{E}(\int_0^T |b(0,\alpha_t)|^2 + |\sigma(0,\alpha_t)|^2 dt < \infty \}$

ii) $J(t,x) := \mathbb{E}(\int_t^T f(X_s^{t,x}, \alpha(s)) + g(X_T^{t,x}))$

iii) $V(t,x) = \sup_{\alpha \in \mathbb{A}} J(t,x)$.

Thm. (Dynamic Programming Principle)

For $0 \leq t \leq s \leq T$, $x \in \mathbb{R}^d$. We have:

$$V(t,x) = \sup_{\alpha \in \mathbb{A}} \mathbb{E}(\int_t^s f(X_r^{t,x}, \alpha(r)) + V(s, X_s^{t,x}))$$

Pf: 1°) By Markov prop.:

$$J(t,x,\alpha) \leq \mathbb{E}(\int_t^s f(\cdots) + V(\cdots)).$$

2') $\forall \varepsilon > 0$. $w \in \Omega$. $\exists\, \alpha^{\varepsilon, w} \in \mathbb{A}$.

s.t. $V(s, X_s^{t,x}(w)) - \varepsilon \leq J(s, X_s^{t,x}(w), \alpha^{\varepsilon, w})$

Set $\hat{\alpha}^{\varepsilon}_r(w) = \begin{cases} \alpha_r(w), & r \leq s \\ \alpha_r^{\varepsilon, w}(w), & r > s \end{cases}$

And $\tilde{\alpha}^{\varepsilon}$ is modification of $\hat{\alpha}^{\varepsilon}$.

$\Rightarrow V(t,x) \geq J(t, x, \tilde{\alpha}^{\varepsilon})$.

$\overset{\text{mp.}}{\geq} \mathbb{E}\left(\int_t^s f(X_r^{t,x}, \alpha_r)\,dr + V(s, X_s^{t,x})\right) - \varepsilon \quad \forall \varepsilon > 0$

Def: $\varphi: \mathbb{R}^d \to \mathbb{R}^1$, regular enough. Set generator

$L^a \varphi(x) = b(x, a) \cdot \nabla_x \varphi(x) + \frac{1}{2} Tr(\sigma\sigma^T(x, a) \nabla^2 \varphi(x))$

Thm: (Feynman-Kac)

Consider $d\tilde{X}_r^{t,x} = b(\tilde{X}_r^{t,x})\,dr + \sigma(\tilde{X}_r^{t,x})\,dW_r$

If $V(t,x) \in C^{1,2}([0,T], \mathbb{R}^d)$, satisfy:

$\begin{cases} \partial_t V(t,x) + L V(t,x) + f(t,x) = 0. \\ V(T, x) = g(x). \end{cases}$

Then: $V(t,x) = \mathbb{E}\left(\int_t^T f(r, \tilde{X}_r)\,dr + g(\tilde{X}_T^{t,x})\right)$

Pf: Apply Itô's formula on

$V(r, \tilde{X}_r^{t,x})$, and use MP.

Thm. (Verification)

If $v(t,x) \in C^{1,2}([0,T]\times\mathbb{R}^d)$, satisfies

$v(T,x) = g(x)$ and HJB equation:

$\partial_t v(t,x) + \sup_{a\in A} \{ L^a v(t,x) + f(x,a) \} = 0$. Then:

For $\hat{\alpha} : [0,T]\times\mathbb{R}^d \to A$. measurable. st.

$\hat{\alpha}(t,x) \in \arg\max_{a\in A} \{ L^a v + f(x,a) \}$ and $(X_t, \hat{\alpha})$

has solution. $\Rightarrow$ $v(t,x) = V(t,x)$

and $\hat{\alpha}(t,x)$ is optimal control.

Pf: $g(X_T^{t,x}) = V(T, X_T^{t,x})$

$\overset{\text{Itô}}{=} v(t, X_t^{t,x}) + \int_t^T (\cdots)$

Take expectation and use HJB:

$\mathbb{E}(g(X_T^{t,x})) \le v(t,x) - \mathbb{E}\left(\int_t^T f(X_r^{t,x}, \alpha_r) \, dr\right)$

for $\forall \alpha \in \mathcal{A}$

**Def:** Hamilton $H(x,y,z): \mathbb{R}^d \times \mathbb{R}^d \times Sym^{d\times d} \to \mathbb{R} \cup \{\infty\}$

defined by $H(x,y,z) = \sup_{a\in A} \{ b(x,a)\cdot y + \frac{1}{2} Tr(\sigma\sigma^T(x,a)\cdot z) + f(x,a) \}$.

**Rmk:** i) HJB equation can be written in: $\partial_t V(t,x) + H(x, \nabla V(t,x), \nabla^2 V) = 0$ which is parabolic PDE

First solve $\hat{a} \in \arg\sup\{\cdots\}$ and obtain $H(x,y,z)$

ii) If $\sigma$ isn't controlled. Then it's semilinear PDE

iii) Note $H$ can be $+\infty$. The verif. Thm can be applied if we can find $V(t,x)$ avoids the bad point at $H=\infty$.

**e.g.** (Infinite horizon problem)

Consider $\sup_{\alpha} \mathbb{E}(\int_0^\infty e^{-\beta t} f(X_t^x, \alpha_t)dt)$

Apply Itô on $e^{-\beta t} w(X_t^x)$. We know the HJB equation should be:

$-\beta w(x) + \sup_a \{\cdots + f(x,a)\} = 0$, $\lim_{T\to\infty} e^{-\beta T} \mathbb{E}(w(X_T^x)) = 0$