Static MFG.

- Static games means they has no time component.

(1) Limit Thms:

Def: Action space A is cpt metric

i) Strategy profile is vector $(a_1 \cdots a_n) \in A^n$.

ii) Objective func. of player i in n-player game is $J_i^n : A^n \to \mathbb{R}^1$.

Rmk: Player i's goal is to choose a_i to maxize J_i^n.

iii) Nash equilibrim for n-player is $(a_1 \cdots a_n) \in A^n$. s.t.

$J_i^n(a_1 \cdots a_n) \geq J_i^n(a_1, \cdots \tilde{a}, a_{i+1} \cdots a_n)$

$\forall i \leq n$. $\forall \tilde{a} \in A$.

if for given $\varepsilon \geq 0$. we have:

$$J_i^n(a_1, \cdots a_n) \geq J_i^n(a_1, \cdots \tilde{a}, a_{i+1}, \cdots a_n) - \varepsilon.$$

then we call it ε-Nash equi.

<u>Rmk</u>: Nash equi means each player i is choosing a_i opt. given the other players.

iv) Payoff function $F: A \times P(A) \to \mathbb{R}^1$.

set $J_i^n(a_1, \cdots a_n) \triangleq F(a_i, \frac{1}{n}\sum_1^n \delta_{a_i})$

<u>Rmk</u>: The structure is symmetric i^{th} is only a label. So the name of this game is called anonymous.

Next, we assume $F: A \times P(A) \to \mathbb{R}^1$ is joint conti. w.r.t. (a, W_1)

① **Thm.** If for each n, given $\varepsilon_n \geq 0$.

and ε_n-Nash equi. $(a_1^n, \cdots a_n^n)$.

s.t. $\lim_{n\to\infty} \varepsilon_n = 0$. Denote $\mu^n = \sum_1^n \delta_{a_i^n}/n$.

Then: i) (μ_n) is tight in $\mathcal{P}(A)$

ii) For $\forall$ weak limit μ. $\mu\{a \in A \mid$

$F(a, \mu) = \sup_{b \in A} F(b, \mu)\} = 1$.

Pf: Note A is cpt metric.

$\Rightarrow (\mu_n)$ is pre-cpt $\Rightarrow$ tight.

Set $\mu_n^i(b) := \frac{1}{n}(\delta_b + \sum_{k\neq i}^n \delta_{a_k^n})$

Note we have: $\forall i$, $\forall b \in A$.

$F(a_i^n, \mu_n) \geq F(b, \mu_n^i(b)) - \varepsilon_n$.

$$\overset{\text{Average}}{\Longrightarrow} \int_A F(a, \mu_n)\, \mu_n(da) \geq \frac{1}{n}\sum_1^n F(b, \mu_n^i(b)) - \varepsilon_n.$$

$\downarrow n_k \to \infty$

$\int_A F(a, \mu)\, \mu(da)$. by joint conti.

combine with the following:

$$W_1(\mu_n, \mu_n^i(b)) \overset{\pi = \sum_{k\neq i} \delta_{(\hat a_k, \hat a_k)} + \delta_{(\hat a_i, b)}}{\le} d(\hat a_i, b)/n \overset{A \text{ cpt}}{\le} c/n.$$

$$\Rightarrow \lim_{n\to\infty} \max_{b\in A} \max_{i} |F(b, \mu_n^i(b)) - F(b, \mu_n)| \overset{\text{Lip.}}{=} 0$$

S_3: Set $\varepsilon_k \to 0$. We have:

$$\int_A F(a, \mu)\, d\mu(a) \geq F(b, \mu). \quad \forall b \in A.$$

i.e. $\int_A F(a, \mu)\, \mu(da) \geq \sup_b F(b, \mu)$

<u>Rmk</u>: Existence of (ε-) Nash equi. may not hold in pure strategy. But it's possible when working with mixed strategy.

<u>Def</u>: $\mu \in \mathcal{P}(A)$ is called MFE (mean field equilibrium) if $\mu(\{a \in A \mid F(a, \mu) = \sup_b F(b, \mu)\}) = 1$.

<u>Rmk</u>: The THM shows the limit points of Nash equi. must be MFE.

② **Thm.** (Uniqueness)

If F satisfies the monotonicity condition

$$\int_A (F(a,m_1)-F(a,m_2))(m_1-m_2)(da) < 0.$$

for $\forall m_1 \neq m_2 \in P(A)$. Then: There's at most one MFE.

Rmk: It's not enough that:

$a \mapsto F(a,m)$ has unique max point for each m. Since:

$(\delta_{\hat{a}(m)})_{m\in P(A)}$ ✓. where $(\hat{a}(m))$ is set of maximizer. for m.

Pf: If m_1, m_2 are both MFE.

$$\overset{i,j=1,2}{\Rightarrow} \int_A F(a,m_i)m_i(da) - \int F(a,m_i) m_j(da) \geq 0.$$

So: $$\int_A (F(a,m_1)-F(a,m_2))(m_2-m_1)(da) \geq 0.$$

③ **Thm** (Converse of Limit THM)

If $\mu \in P(A)$ is a MFE. Then:

$\exists\ (\varepsilon_n) \geq 0$. and seq of strategy

profiles $(a_k)_{k \geq 1} \subset A$. s.t. $(a_1 \cdots a_n)$

ε_n-nash equi. for n-player and

$\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_1^n \delta_{a_i} = \mu$. $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Pf: $X_k \overset{i.i.d}{\sim} \mu$. A-valued r.v.'s.

defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Set $\mu_n := \frac{1}{n} \sum\limits_1^n \delta_{X_k}$. and

$\mu_n^{(i)}(a) := \frac{1}{n}(\delta_a + \sum\limits_{k \neq i}^n \delta_{X_k})$.

Note $\mu_n \xrightarrow{w} \mu$. a.s. Next.

prove: $\varepsilon_n \triangleq \max\limits_{i \leq n} (\sup\limits_{a \in A} F(a, \mu_n^{(i)}(a)) - F(X_i, \mu_n))$

$\longrightarrow 0$. a.s.

Set $\tilde{\varepsilon}_n := \max\limits_{i \leq n} (\sup\limits_{a \in A} F(a, \mu_n) - F(X_i, \mu_n))$

Note by uniform conti. of $F(a,\cdot)$. (A cpt.)

we have $\lim_{n\to\infty} |\Sigma_n - \tilde{\Sigma}_n| = 0$. a.s.

Again $\lim_{n\to\infty} |\tilde{\Sigma}_n - \max_{i\le n} (\underbrace{\sup_{a\in A} F(a,m) - F(x_i,m)}_{=0})| = 0$.

Since X_i support on M. $\Rightarrow$ 0. a.s.

Rmk: If we require: $\forall n$. $\Sigma_n = 0$. then the theorem won't hold.

e.g. $A = [0,1]$. $F(a,m) := a\int_{[0,1]} x\, m(dx)$

$\Rightarrow$ MFE $= \delta_0, \delta_1$

The only Nash equi. is:

$(a_1, \cdots a_n)$. $\forall k$. $a_k = 1$.

But $(0, 0 \cdots 0)$ is a $\frac{1}{n}$-Nash equi. $\to (0, 0 \cdots 0 \cdots)$.

④ Thm (Existence)

There always exists a MFE.

Pf: **Lemma** (Kakutani) If K is convex cpt. subset of locally cpt. t.v.s. and $\Gamma: K \to 2^K$. (set-valued). s.t.

i) $\Gamma(x)$ is nonempty. convex. $\forall x$.

ii) Graph $Gr(T) = \{(x,y) \in K \times K.\ y \in T(x)\}$ is closed.

Then: There exists a fixed point. i.e. $x \in T(x)$

Set: $T: P(A) \longrightarrow P(A)$. define by

$$T(\eta) := \{ M \in P(A) \mid M\{a \in A \mid F(a, \underset{\sim}{M}) = \sup_{b \in A} F(b, \underset{\sim}{M}) = 1\}.$$

Next, we check the conditions of Lem.

i) $P(A)$ is cpt. convex. of $M(A)$. the space of signed measures.

2') $S = \{a \in A \mid F(a,\mu) = \sup_A F(b,\mu)\} \neq \emptyset$.

Since F is conti.

$\Rightarrow T(\mu) \neq \emptyset$. for $\forall \mu \in P(A)$.

And $T(\mu)$ is convex is trivial.

3') $Gr(T) = \bigcap_{b \in A} K_b$. where we denote

$K_b := \{(\mu, m) \in P(A)^{\otimes 2} \mid \int F(a,\mu) m(da) \geq F(b,\mu).\ \forall b \in A\}$.

prove K_b is close by limit point argument. follows from F is conti.

(2) <u>Multiple Types of Agents:</u>

Next, we introduce more general case

<u>Def</u>: Type space $\mathcal{T}$ is polish

i) Payoff func. $F: \mathcal{T} \times A \times P(\mathcal{T} \times A) \longrightarrow \mathbb{R}^1$.

And the objective function is:

$$J_i^n(a_1 \cdots a_n) \triangleq F(t_i, a_i, \frac{1}{n}\sum_1^n \delta_{(t_k, a_k)})$$

ii) Let $C: J \longrightarrow 2^A$. set-value map. called constraint map. Denote graph of C by $Gr(C) := \{(t, a) \in J \times A.\ a \in C(t)\}$.

iii) ε-Nash equi. associated with $(t_1 \cdots t_n)$ is $(a_1 \cdots a_n) \in A^{\otimes n}$. s.t. $\forall i.\ a_i \in C(t_i)$ and $\forall b \in C(t_i)$. :

$$J_i^n(a_1 \cdots a_n) \geq J_i^n(a_1 \cdots b, a_{i+1} \cdots) - \varepsilon$$

<u>Thm.</u> (Berge's)

If F and C satisfy the followings:

i) F is joint conti on $Gr(C) \times \mathcal{P}(Gr(C))$.

ii) $\forall t \in J.\ C(t) \neq \emptyset$. iii) $Gr(C)$ is closed.

iv) C is hemiconti. i.e. if $t_k \to t$ in J. and $a \in C(t)$. $\Rightarrow \exists (k_j)$. $a_j \in C(t_{k_j}) \to a$.

Them: $C^*(t,m) = \{a \in C(t) \mid F(a,t,m) = \sup_{b \in C(t)} F(t,b,m)\}$ is closed and

$F^*(t,m) = \sup_{b \in C(t)} F(t,b,m)$ is joint conti.

Pf: $C^*(t,m) = \bigcap_{b \in C(t)} \{F(t,a,m) \geq F(t,b,m)\}$

And conti. follows from $C(t)$ is pre-cpt and joint conti. of F.

Cor. Under the assumptions above, If given $(t_1^n, \dots t_n^n) \in \mathcal{T}^{\otimes n}$, ε_n-Nash equi. $(a_1^n \dots a_n^n) \in A^{\otimes n}$ for the corresponding game. st. $\varepsilon_n \to 0$. and $\lim_{n \to \infty} \frac{1}{n} \sum_1^n \delta_{t_k^n} = \lambda \in \mathcal{P}(\mathcal{T})$. Then:

$\mu_n := \frac{1}{n} \sum_1^n \delta_{(t_k^n, a_k^n)}$ is tight and

$\forall$ weak limit point $\mu \in \mathcal{P}(\mathcal{T} \times A)$ satisfies $\mu\{(t,a) \in \mathcal{T} \times A \mid F(t,a) = F^*(t,\mu)\} = 1$

Pf: Note A is cpt. Set $M = \overline{\bigcup_{k,n} \{a_k^n\}}$

and $\overline{\lim_{n\to\infty}} \frac{1}{n}\sum_1^n \delta_{t_k^n}(N) \geq 1-\varepsilon$.

Since it's tight. Then let

$P = N \otimes M$ is the set to prove:

$\frac{1}{n}\sum_1^n \delta_{(t_k^n,\, a_k^n)}$ is tight.

The latter is argued as before

Cor. Define MFE as Thm above.

There's always a MFE.

Pf: By Berge's Thm. argue as before.

eg. (Congestion game)

$G=(V,E)$. finite directed graph

Set $J = V \times V$. set of source-destination pair. And set A is set of Hamilton paths of G.