Stochastic Diff. Game

(1) Two players games:

Def: If we have 2 players on action sets A, B, and objective function $F: A \times B \to \mathbb{R}^1$.

i) It's zero-sum game if A is to maxize F and B is to max-mize $-F$

ii) Nash equi. in such zero-sum game is $(a^*, b^*) \in A \times B$, s.t.

$$\inf_b F(a^*, b) = F(a^*, b^*) = \sup_a F(a, b^*).$$

Rmk: If such Nash equilibrim exists. Then, we have:

It's called value of the game $\leftarrow$ $F(a^*,b^*) = \sup_a \inf_b F(a,b) = \inf_b \sup_a F(a,b)$

since $\inf_b F(a^*,b) \leq \sup_a \square \cdots \sup_a \cdots$.

i.e. it means nash equi. is a saddle point of F.

Next, we suppose the two players α, β. control a d-dim process X:

$dX_t = b(X_t, \alpha_t, \beta_t)\,dt + \sigma(X_t, \alpha_t, \beta_t)\,dW_t$.

And the objective is:

$J(\alpha,\beta) := \mathbb{E}\left(\int_0^T f(X_t,\alpha_t,\beta_t)\,dt + g(X_T)\right)$.

Where $\alpha, \beta : [0,T) \times \mathbb{R}^d \to A, B$, are Markovian control. / close loop control. (i.e. $\alpha = \alpha(t, X_t)$.)

<u>Rmk</u>: i) If in the case of one player there's no difference whether we use Markovian control or open control. (measurable). which can be

represented by $(t, X_t(\omega))$.

ii) There's another kind Markovian control : $\tau(\omega) = \alpha(t, (X_t(\omega))_t)$. which's path - dependent

<u>Def</u>: The game has value if $\sup_\alpha \inf_\beta J(\alpha, \beta)$

$= \inf_\beta \sup_\alpha J(\alpha, \beta)$.

<u>To optimize</u>: Fix β, α resp. we can get V^β, V^α satisfy HJB :

$$\partial_t V^\beta + \sup_\alpha h(x, \nabla V^\beta, \nabla^2 V^\beta, \alpha, \beta) = 0$$

$$\partial_t V^\alpha + \inf_\beta h(x, \nabla V^\alpha, \nabla^2 V^\alpha, \alpha, \beta) = 0$$

where $h(x, y, z, \alpha, \beta) = b(x, \alpha, \beta) \cdot y$

$+ \frac{1}{2} Tr(\sigma\sigma^T(x, \alpha, \beta) \cdot z) + f(x, \alpha, \beta)$.

and $V^\alpha(T, x) = V^\beta(T, x) = g(x)$.

Rmk: If (α^*, β^*) is Nash. Then: V^{α^*}, V^{β^*} both satisfy the same PDE:

$$\partial_t V + h(x, \nabla V, \nabla^2 V, \alpha^*, \beta^*) = 0.$$

$\Rightarrow$ By Feynman-Kac representation: we have $V = V^{\alpha^*} = V^{\beta^*}$.

And $\sup_\alpha h(\dots) = \inf_\beta h(\dots)$. i.e. (α^*, β^*) is saddle point of h.

Def. i) $H^+(x, y, z) := \inf_\beta \sup_\alpha h(x, y, z, \alpha, \beta)$.

$H^-(x, y, z) := \sup_\alpha \inf_\beta h(x, y, z, \alpha, \beta)$.

ii) We say Isaac's condition holds if $H^+ \equiv H^-$.

Rmk: i) By Rmk above. Nash equilibrium exists $\Rightarrow$ Isaac's condition holds.

ii) Isaac's condition is key to ensure value exists.

<u>Thm.</u> (Verification)

If Isaac's condition holds ($H \overset{\Delta}{=} H^{\pm}$).

and $\exists\, V \in C^{\infty}$. s.t.

$\partial_t V(t,x) + H(x, \nabla V, \nabla^2 V) = 0$. $V(T,x) = g(x)$.

(α^*, β^*) is measurable saddle point of

$(\alpha, \beta) \longmapsto h(x, \nabla V, \nabla^2 V, \alpha, \beta)$. $\forall (t,x)$.

Besides, the state equation $dX_t =$

$b(X_t, \alpha(t, X_t), \beta(\cdot\cdot)) dt + \cdots$ is well-posed.

Then: (α^*, β^*) is Nash equi. closed loop.

<u>eg.</u> (Buckdahn's)

When we use open loop control.

the value may not exist.

consider $X = (X^1, X^2)$. $W = (W^1, W^2)$.

$$\begin{cases} dX_t^1 = \alpha\, dt + \sigma\, dW_t^1 \\ dX_t^2 = \beta\, dt + \sigma\, dW_t^2 \end{cases} \quad X_0^1 = X_0^2 = x.$$

the value space of control is $A = B = [0,1]$.

Remote objective $J(\alpha,\beta) = \mathbb{E}(|X_T^1 - X_T^2|)$.

$\Rightarrow h(y^1, y^2, z, \alpha, \beta) = \alpha y^1 + \beta y^2 + \frac{1}{2}\sigma^2 (z_1 + z_2)$.

And $H^I(x, y, z) = |y^1| - |y^2|$.

So it satisfies Isaac's condition. But

<u>Thm</u>, $\underline{V}_0 := \sup_\alpha \inf_\beta J < \overline{V}_0 := \inf_\beta \sup_\alpha J$.

if $0 \le \sigma < \frac{1}{2}\sqrt{\pi T}$. in open loops.

<u>Pf</u>: 1°) Set $\alpha = \beta$. $\Rightarrow J(\alpha,\beta) = 2\sigma\sqrt{T/2}$

So: $\underline{V}_0 \le 2\sigma\sqrt{T/2}$.

2°) Set $\tilde{\alpha} = -\frac{\mathbb{E}(X_T^2)}{|\mathbb{E}(X_T^2)|} I_{\{\mathbb{E}(X_T^2) \ne 0\}} + I_{\{\mathbb{E}(X_T^2) = 0\}}$.

$\Rightarrow J(\tilde{\alpha}, \beta) \ge |\mathbb{E}(X_T^1 - X_T^2)|$.

$= |\tilde{\alpha} T + \tilde{\alpha}\,\mathbb{E}|X_T^2|| \ge T$.

So: $\overline{V}_0 \ge T$.

(2) n-player game:

① Consider n players choose control $(\alpha_t^i)_1^n \in A_1 \times \cdots A_n$. to influence:

$dX_t = b(X_t, \vec{\alpha}_t)dt + \sigma(X_t, \vec{\alpha}_t)dW_t$

$b, \sigma : \mathbb{R}^d \times \prod_1^n A_i \to \mathbb{R}^d . \mathbb{R}^{d\times m}$.

Next, we consider the closed loop control to optimize objective:

$J_i(\vec{\alpha}) := \mathbb{E}\left(\int_0^T f_i(X_t, \vec{\alpha}(t, X_t))dt + g_i(X_T)\right)$

<u>Def</u>: i) Closed loop Nash equi.

$\vec{\alpha}$ is: $\forall i$. $\forall \beta \in A_i$. we have:

$J_i(\vec{\alpha}) \geq J_i(\vec{\alpha}^{-i}, \beta)$.

ii) Hamillton for player i is $H_i($

$x, y, z, \vec{\alpha}) = b(x, \vec{\alpha}) \cdot y + \frac{1}{2} Tr(\sigma\sigma^T(x, \vec{\alpha})z) + f_i(x, \vec{\alpha})$

Rmk: Each player will have their own value func. $V_i(t,x)$ as well!

iii) Generalized Isaac's condition holds if $\exists$ measurable $\alpha_i : [0,T] \times \mathbb{R}^d \times (\mathbb{R}^d)^n \times S_d^n \to A_i$ s.t. $\vec{\alpha} := (\alpha^1, \dots, \alpha^n)$ is Nash equi for static n-player game with given func. $\prod_i A_i \ni (a_1, \dots, a_n) \mapsto h_i(x, y_i, z_i, a_1, \dots, a_n)$. $\forall (x, \vec{y}, \vec{z})$.

Theorem 7.3 (Verification theorem). *Suppose the generalized Isaacs' condition holds. Suppose $\vec{v} = (v_1, \dots, v_n)$, with $v_i : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ for each i, is a $C^{1,2}$ solution of the PDE system*

$$\partial_t v_i(t,x) + h_i\Big(x, \nabla v_i(t,x), \nabla^2 v_i(t,x), \vec{\alpha}(x, \nabla \vec{v}(t,x), \nabla^2 \vec{v}(t,x))\Big) = 0,$$

$$v_i(T,x) = g_i(x),$$

where we abbreviate $\nabla \vec{v} = (\nabla v_1, \dots, \nabla v_n)$ and $\nabla^2 \vec{v} = (\nabla^2 v_1, \dots, \nabla^2 v_n)$. Finally, setting $\vec{\alpha}^*(t,x) = \vec{\alpha}(x, \nabla \vec{v}(t,x), \nabla^2 \vec{v}(t,x))$, suppose that the state equation

$$dX_t = b(X_t, \vec{\alpha}^*(t, X_t))dt + \sigma(X_t, \vec{\alpha}^*(t, X_t))dW_t$$

is well-posed. Then $\vec{\alpha}^*$ is a closed loop Nash equilibrium.

Pf: Note $\partial_t V_i + \sup_{A_i} h_i(x, \nabla V_i, \cdots) = 0$

by def of $\vec{\alpha}$. (Isaac's cond.)

Apply Verification Thm on one

player. $\Rightarrow V_i(t,x) = \sup_{\alpha_i} \mathbb{E}\left(\int_t^T\right.$

$f_i(X_s^{t,x}, (\vec{\alpha}^*(s, X_s))^{-i}, \alpha_i) dt + g(X_T^{t,x}))$

where $dX_s^{t,x} = b(X_s^{t,x}, (\vec{\alpha}^*(s, X_s^{t,x}))^{-i},$

$\alpha^i)) ds + \sigma(\cdots) dW_s$.

By Isaac's cond. $\Rightarrow \alpha_i^*$ is point-

wise optimizer. So it's truly op-

timal control. repeat in i.

② Private State process:

Consider $d = n = m+1$. And dynamics:

$dX_t^i = b_i(\vec{X}_t, \alpha_t^i) dt + \sigma_i(\vec{X}_t, \alpha_t^i) dW_t^i + \hat{\sigma}_i(\vec{X}_t) dB_t$

Set Objective is:

$J_i(\vec{\alpha}) = \mathbb{E}\left(\int_0^T f_i(\vec{X}_t, \alpha_t^i) dt + g_i(\vec{X}_T)\right)$

Set the Hamiliton of i^{th} is:

$$h_i(x, y, z, \vec{a}) = \sum_1^n b_k(x, a_k) y_k + \frac{1}{2}\sum_1^n \sigma_k^2(x, a_k) z_{kk} + \frac{1}{2}\sum_{k,j}^n \tilde{\sigma}_k(x)\hat{\sigma}_j(x) z_{kj} + f_i(x, a_i)$$

Rmk: Set $\tilde{h}_i(x, y, z, a_i) = b_i(x, a_i) y_i + \frac{1}{2}\sigma_i^2(x, a_i) z_{ii} + f_i(x, a_i)$

$\Rightarrow h_i - \tilde{h}_i(\dots a_i) \stackrel{\Delta}{=} h_i'$

is indept of a_i.

Lemma. For each i. if $\exists$ measurable α_i $: [0,T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to A_i$. s.t. $\forall (x, y, z)$.

We have: $\alpha_i(x, y, z) \in \underset{A_i}{\arg\max}\, \tilde{h}_i(\dots, a_i)$

$\Rightarrow$ Isaac's condition holds.

$\Rightarrow$ Set optimize Hamilition $H_i(x, y, z) = \sup_{A_i} \tilde{h}_i(\dots, a_i)$

We have HJB system:

$$0 = \partial_t V_i(t, x) + H_i(x, \partial_{x_i} V_i, \partial_{x_i x_i} V_i) + h_i'(x, \partial_{x_i} V_i \cdots)$$

$$V_i(T, x) = g_i(x). \quad \text{for } \alpha_i^*$$