The Master Equation

Next we introduce an analytic approach to MFG.

(1) <u>Calculus on $P(\mathbb{R}^d)$:</u>

<u>Def</u>: i) $U : P(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if $\exists$
$\frac{\delta U}{\delta m} \in C_B(P(\mathbb{R}^d) \times \mathbb{R}^d ; \mathbb{R}^d)$. s.t. for $\forall m, m' \in P(\mathbb{R}^d)$. We have:

$$U(m) - U(m') =$$

$$\int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-t)m + tm', v)(m'-m)(dv)dt$$

<u>Rmk</u>: i) Equivalently, we can def:

$$\frac{d}{dh}\Big|_{h=0} U(m' + h(m-m')) =$$

$$\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m, v)(m-m')dv$$

Pf: ($\Rightarrow$) Note: $\frac{d}{dh}\Big|_{h=0} D = \lim_{h\to 0} \frac{u(m+h(m'-m)) - u(m)}{h}$

Set $m' = m + h(m'-m)$ on Def i)

let $t = ht$. divide h

on both sides. let $h \to 0$

($\Leftarrow$) Set $m = (1-t)\tilde{m} + t m'$

divide $(1-t)$ on both

sides (on Rmk). Then:

$$LHS = \lim_{u\to 0} \frac{u(m' + ((1-t)-u)(\tilde{m}-m')) - u(m' + (1-t)(\tilde{m}-m'))}{u}$$

where we let $x = (1-t)h$

$$\Rightarrow LHS = \frac{d}{dx}\Big|_{x=1-t} u(m' + x(\tilde{m}-m'))$$

integrate t from 0.1. on

both sides.

ii) $\frac{\delta u}{\delta m}(m,v) + c(m)$ also satisfies the condition, $\forall c(m)$.

We choose: $\int \frac{\delta u}{\delta m}(m,v)\, m(dv) = 0$.

ii) If $\frac{\delta u}{\delta m}(m,v) \in C^1(\mathbb{R}^d)$. We define

$D_m u(m,v) = D_v\left(\frac{\delta u}{\delta m}(m,v)\right)$. And

$D_v D_m u(m,v) \stackrel{\Delta}{=} D_v(D_m u(m,v)) \in \mathbb{R}^{d\times d}$.

iii) u is $\mathcal{C}^2$. if $m \mapsto \frac{\delta u}{\delta m}(m,v) \in \mathcal{C}^1$.

denote $\frac{\delta^2 u}{\delta m^2}(m,v,v') := \frac{\delta}{\delta m}\left(\frac{\delta u}{\delta m}(m,v)\right)(v')$

Also, $D_m^2 u(m,v,v') := D^2_{v,v'} \frac{\delta^2 u}{\delta m^2}(m,v,v')$.

<u>Lemma.</u> $u(x,m) : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^1$. We have:

$D_x D_m u(x,m,v) = D_m D_x u(x,m,v)$

if both sides exist and bdd.

Pf: Apply D_x on both sides of Def i), we have:

$$D_x U(x,m') - D_x U(x,m) = \int_0^1 \int_{\mathbb{R}^d} D_x \frac{\delta U}{\delta m}(x, tm' + (1-t)m, v)(m'-m)(dv).$$

So: $\frac{\delta}{\delta m}(D_x U) = D_x(\frac{\delta}{\delta m} U)$.

$\Rightarrow$ Apply D_v on both sides.

Lemma. If $U \in \mathcal{C}^2$. Then: we have.

$$D_m^2 U(m,v,v') = D_m(D_m U(m,v))(v')$$

Pf: By Lemma above and Def iii)

prop. $\forall \vec{x} \in (\mathbb{R}^d)^n$. Set $m_{\vec{x}}^n := \sum_1^n \delta_{x_k}/n$. given $U: P(\mathbb{R}^d) \to \mathbb{R}^1$. $U_n(\vec{x}) := U(m_{\vec{x}}^n)$. Then:

i) If $U \in \mathcal{C}^1$. $D_m U \in C_B$. exists.

Then: $u_n \in C^1$. and $D_{x_j} u_n(\vec{x}) = \frac{1}{n} \cdot$

$D_m u(m^n_{\vec{x}}, x_j)$. $\forall j \leq n$.

ii) If $u \in C^2$. $D_m^2 u \in C_B$. exists. Then:

$u_n \in C^2$. and $D_{x_k} D_{x_j} u_n(\vec{x}) = \delta_{jk} \frac{1}{n} D_v D_m u(m^n_{\vec{x}}, x_j)$

$+ \frac{1}{n^2} D_m^2 u(m^n_{\vec{x}}, x_k, x_j)$

Pf: WLOG. $x_i \neq x_j$. Define $\phi(x_i) = v$

$\phi(x_j) = 0$. $\forall j \neq i$. $\vec{v} = (0, \cdots \overset{j^{th}}{v}, \cdots 0) \in (\mathbb{R}^d)^n$

$$\lim_{h \to 0} \frac{u_n(\vec{x} + h\vec{v}) - u_n(\vec{x})}{h} =$$

$$\lim_{h \to 0} \frac{u(m^n_{\vec{x}} \circ (id + h\phi)) - u(m^n_{\vec{x}})}{h}$$

Note: $u(m^n_{\vec{x}} \circ (\cdots)) - u(m^n_{\vec{x}})$

$$\overset{\text{Def i)}}{=} \int_0^1 \int_{\mathbb{R}^d} \frac{\delta u}{\delta m}(\square, v)(m^n_{\vec{x}} \circ (id + h\phi) - m)(dv)\,dt$$

$$\underset{\text{varia.}}{\overset{\text{change}}{=}} \int_0^1 \int_{\mathbb{R}^d} \left(\frac{\delta u}{\delta m}(\square, v + h\phi(v)) - \frac{\delta u}{\delta m}(\square, v) \right) m(dv)\,dt.$$

$= h\int_0^1\int_0^1\int_{\mathbb{R}^d} D_m U(\square,\ v+sh\phi(v))\,\phi(v)\,ds$
$m(dv)\,dt$

$\overset{DCT}{\Longrightarrow}$ LHS $= \int_{\mathbb{R}^d} D_m U(m^n_{\vec{x}},\ v)\,\phi(v)\,m^n_{\vec{x}}(dv)$

$= \frac{1}{n}\sum_1 D_m U(m^n_{\vec{x}},\ x_k)\,\phi(x_k)$

$= \frac{1}{n} D_m U(m^n_{\vec{x}},\ x_j)\cdot v.$

As for ii), repeat the procedure and use the Lemma above

(2) <u>Itô Formula:</u>

<u>Thm.</u> For $X, \bar{X}$ satisfy SDEs:

$dX_t = b(X_t, \mu_t)\,dt + \sigma(X_t, \mu_t)\,dW_t +$
$\gamma(X_t, \mu_t)\,dB_t,\quad X_0 = \xi.$

$d\bar{X}_t = \bar{b}(\bar{X}_t, \mu_t)\,dt + \bar{\sigma}(\bar{X}_t, \mu_t)\,dW_t +$
$\bar{\gamma}(\bar{X}_t, \mu_t)\,dB_t,\quad \bar{X}_0 = \bar{\xi}.$

where $\mu_t \triangleq \mathcal{L}(\bar{X}_t \mid \mathcal{F}^B) = \mathcal{L}(\bar{X}_t \mid \mathcal{F}^B_t)$

If all the coefficients are bdd uniformly Lip. on X-var and conti. on m-var.

Then for $F : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ smooth we have

$$dF(X_t,\mu_t) = \underbrace{[D_xF(X_t,\mu_t)b(X_t,\mu_t) + \frac{1}{2}\mathrm{Tr}[D_x^2F(X_t,\mu_t)(\sigma\sigma^T+\gamma\gamma^T)(X_t,\mu_t)]]}_{\triangleq LF(X_t, m_t)}dt$$

$$+\int_{\mathbb{R}^d} D_mF(X_t,\mu_t,v)\bar{b}(v,\mu_t)\mu_t(dv)dt$$

$$+\int_{\mathbb{R}^d}\frac{1}{2}\mathrm{Tr}[D_vD_mF(X_t,\mu_t,v)(\bar\sigma\bar\sigma^T+\bar\gamma\bar\gamma^T)(v,\mu_t)]\mu_t(dv)dt$$

$$+\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{2}\mathrm{Tr}[D_m^2F(X_t,\mu_t,v,\tilde v)\bar\gamma(v,\mu_t)\bar\gamma^T(\tilde v,\mu_t)]\mu_t(dv)\mu_t(d\tilde v)dt$$

$$+\int_{\mathbb{R}^d}\mathrm{Tr}[D_xD_mF(X_t,\mu_t,v)\gamma(X_t,\mu_t)\bar\gamma^T(v,\mu_t)]\mu_t(dv)dt$$

$$+\,dM_t,$$

lot-part. UΔ

$m_t \leftarrow \bar{L}F(X_t, m_t)$ generator!

where M_t is a martingale given by

$$dM_t = D_xF(X_t,\mu_t)\Big(\sigma(X_t,\mu_t)dW_t + \gamma(X_t,\mu_t)dB_t\Big)$$
$$+\int_{\mathbb{R}^d} D_mF(X_t,\mu_t,v)\bar\gamma(v,\mu_t)\mu_t(dv)dB_t.$$

Pf: We use approxi. method:

Consider $m_t^n = \sum_1^n \delta_{\bar X_t^k}/n$. where $\bar X_t^k$ is copy of $\bar X_t$.

Note $\bar X_t^k$ is i.i.d condition on B.

So $m_t^n \to m$ in pr.

Apply Itô's formula on:

$g(X_t, \bar X_t^1, \cdots \bar X_t^n) \triangleq F(X_t, m_t^n)$.

where we use prop. in (1) to obtain the differentiations.

$\Rightarrow$ Use DCT to let $n \to \infty$.

Rmk: When $b = \bar{b}$. $\sigma = \bar{\sigma}$. $\gamma = \bar{\gamma}$. $\xi = \bar{\xi}$. $\Rightarrow X = \bar{X}$. The Itô formula describes dynamics of Func. on (X_t, μ_t).

Next, we introduce Feynmann-Kac Thm:

Theorem 9.11. *Let $(t, x, m) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$, and consider processes X and $\bar{X}$ that solve the SDEs*

$$dX_s^{t,x,m} = b(X_s^{t,x,m}, \mu_s^{t,m})ds + \sigma(X_s^{t,x,m}, \mu_s^{t,m})dW_s + \gamma(X_s^{t,x,m}, \mu_s^{t,m})dB_s,$$
$$d\bar{X}_t^{t,m} = \bar{b}(\bar{X}_s^{t,m}, \mu_s^{t,m})ds + \bar{\sigma}(\bar{X}_s^{t,m}, \mu_s^{t,m})dW_s + \bar{\gamma}(\bar{X}_s^{t,m}, \mu_s^{t,m})dB_s,$$

on $s \in (t, T]$, with initial conditions $X_t^{t,x,m} = x$ and $\bar{X}_t^{t,m} \sim m$, and with $\mu_s^{t,m} = \mathcal{L}(\bar{X}_s^{t,m} \mid (B_r - B_t)_{r \in [t,s]})$. Define the value function

$$V(t,x,m) = \mathbb{E}\left[g(X_s^{t,x,m}, \mu_s^{t,m}) + \int_t^T f(X_s^{t,x,m}, \mu_s^{t,m})ds\right],$$

for some given nice functions g and f. Suppose $U : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is smooth and satisfies

$$\partial_t U(t,x,m) + LU(t,x,m) + \bar{L}U(t,x,m) + f(x,m) = 0$$
$$U(T,x,m) = g(x,m).$$

Then $U \equiv V$.

Pf: Apply Itô's formula on $u(X_t, \mu_t)$

We have:

$g(X_T, \mu_T) - u(t, x, m) = \int_t^T (\partial_t \cdots) - \square$

Take expectation and use the condition

(3) Verification Thm:

Let $(t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$, and consider the controlled processes X and $\bar{X}$ that solve the SDEs

$$dX_s^{t,x,m} = b(X_s^{t,x,m}, \mu_s^{t,m}, \alpha_s)ds + \sigma(X_s^{t,x,m}, \mu_s^{t,m}, \alpha_s)dW_s + \gamma(X_s^{t,x,m}, \mu_s^{t,m})dB_s,$$
$$d\bar{X}_t^{t,m} = \bar{b}(\bar{X}_s^{t,m}, \mu_s^{t,m})ds + \bar{\sigma}(\bar{X}_s^{t,m}, \mu_s^{t,m})dW_s + \bar{\gamma}(\bar{X}_s^{t,m}, \mu_s^{t,m})dB_s,$$

on $s \in (t, T]$, with initial conditions $X_t^{t,x,m} = x$ and $\bar{X}_t^{t,m} \sim m$, and with $\mu_s^{t,m} = \mathcal{L}(\bar{X}_s^{t,m} \mid (B_r - B_t)_{r \in [t,s]})$. Define the value function

$$V(t, x, m) = \sup_{\alpha} \mathbb{E}\left[g(X_s^{t,x,m}, \mu_s^{t,m}) + \int_t^T f(X_s^{t,x,m}, \mu_s^{t,m}, \alpha_s)ds\right],$$

for some given nice functions g and f. Define the Hamiltonian

$$H(x, m, y, z) = \sup_{a \in A}\left[y \cdot b(x, m, a) + \frac{1}{2}\mathrm{Tr}[z(\sigma\sigma^\top + \gamma\gamma^\top)(x, m, a)] + f(x, m, a)\right].$$

Theorem 9.12. *Suppose $U : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is smooth and satisfies*

$$\left.\begin{aligned} &\partial_t U(t, x, m) + H(x, m, D_x U(t, x, m), D_x^2 U(t, x, m)) + \bar{L}U(t, x, m) = 0 \\ &U(T, x, m) = g(x, m). \end{aligned}\right] (*)$$

Suppose also that there exists a measurable function $\alpha : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to A$ such that $\alpha(t, x, m)$ attains the supremum in $H(x, m, D_x U(t, x, m), D_x^2 U(t, x, m))$ for each (t, x, m) and also the SDE

$$dX_t = b(X_t, \mu_t, \alpha(t, X_t, \mu_t))dt + \sigma(X_t, \mu_t, \alpha(t, X_t, \mu_t))dW_t + \gamma(X_t, \mu_t)dB_t$$

is well-posed. Then $U \equiv V$, and $\alpha(t, X_t, \mu_t)$ is an optimal control.

Pf: As above. apply Itô's on $u(X_t, m_t)$ and take expectation:

$\mathbb{E}(g(X_T, m_T)) - u(t, x, m) =$

$\mathbb{E}(\int_t^T \partial_t u \cdots + \mathcal{L} u \cdots) \leq$

$-\mathbb{E}(\int_t^T f(X_s, m_s, \alpha_s) ds), \ \forall \alpha$

So: $u \geq V$. $\Rightarrow u \equiv v$.

Cor. Set $\bar{\gamma} = \gamma$, $\bar{b}(x, m) = \hat{b}(x, m, D_x u(t, x, m), D_x^2 u(t, x, m))$, $\bar{\sigma}(x, m) = \hat{\sigma}(\sim)$, where $\hat{b}(x, m, y, z) \triangleq b(x, m, \alpha(x, m, y, z))$, $\hat{\sigma}(x, m, y, z) \triangleq b(\sim)$. Then we have $(*)$ called master equation.

Rmk: The master equation is also nonlocal and nonlinear.