MFG with Common Noise

① Assump. A (Lip. and integrable)

Assumption A. The main results of the paper will be proved under the following assumptions, which we assume to hold throughout the paper:

(A.1) A is a closed subset of a Euclidean space. (More generally, as in [21], a closed σ-compact subset of a Banach space would suffice.)

(A.2) $p' > p \geq 1 \vee p_\sigma$, $p_\sigma \in [0,2]$, and $\lambda \in \mathcal{P}^{p'}(\mathbb{R}^d)$. (Here $a \vee b := \max(a,b)$.)

(A.3) The functions b, σ, σ_0, f, and g of (t,x,μ,a) are jointly measurable and are continuous in (x,μ,a) for each t.

(A.4) There exists $c_1 > 0$ such that, for all $(t,x,y,\mu,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}^p(\mathbb{R}^d) \times A$,

$$|b(t,x,\mu,a) - b(t,y,\mu,a)| + |(\sigma,\sigma_0)(t,x,\mu) - (\sigma,\sigma_0)(t,y,\mu)| \leq c_1|x-y|,$$

and

$$|b(t,0,\mu,a)| \leq c_1\left[1 + \left(\int_{\mathbb{R}^d}|z|^p\mu(dz)\right)^{1/p} + |a|\right],$$

$$|\sigma(t,x,\mu)|^2 + |\sigma_0(t,x,\mu)|^2 \leq c_1\left[1 + |x|^{p_\sigma} + \left(\int_{\mathbb{R}^d}|z|^p\mu(dz)\right)^{p_\sigma/p}\right] x.$$

(A.5) There exist $c_2, c_3 > 0$ such that, for each $(t,x,\mu,a) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}^p(\mathbb{R}^d) \times A$,

$$-c_2\left(1 + |x|^p + \int_{\mathbb{R}^d}|z|^p\mu(dz)\right) \leq g(x,\mu) \leq c_2\left(1 + |x|^p + \int_{\mathbb{R}^d}|z|^p\mu(dz)\right),$$

$$-c_2\left(1 + |x|^p + \int_{\mathbb{R}^d}|z|^p\mu(dz) + |a|^{p'}\right) \leq f(t,x,\mu,a) \leq c_2\left(1 + |x|^p + \int_{\mathbb{R}^d}|z|^p\mu(dz)\right) - c_3|a|^{p'}.$$

$\Downarrow$

Thm. Weak solution with weak control exists.

Pf: Discretization and approxi.

1) Discretize $\mathcal{G}_t^B = \sigma(\bigcup_i \mathcal{G}_t^{\tilde{n}})$,

On $\mathcal{G}_t^{n}$ (finite σ-algebra.

Obtain $\mu^n = \mathbb{P}\circ X\in\cdot\,|\,\mathcal{G}_t^n)$ the MFE of discretized optimal. prob.

2) Consider strengthened assump. B

a.) underlying space is cpt.

b.) efficiencies are uniformly bdd.

and $(\bar{\mu}^n := \underbrace{\mathbb{P}_n}\circ(W,\Lambda,X)\in\cdot\,|\,\mathcal{G}_t^n))$ solution. w.r.t. μ^n

the extended p.m.'s.

We have $(\bar{\mu}^n)$ relative cpt.

by Aldous ... And admits weak limit is a MFE pre-solution.

Then, prove it's truly optimal.

So it's weak limit.

3') Replace assump. B. by cpt. approxi. with cpt seq. argument.

⑦ Assump. A + Assump. C:

convex cond. on coefficients

$\Downarrow$

Thm. weak MFE with weak strict control. exists.

⑧ Assump. A. + Assump. D:

variable-separating and convex. cond. on coefficients.

$\Downarrow$

Thm. Weak MFE with strong control. exists.

④ Assmp. A. + Assmp. E:

Lasry-Lions Monotonicity cond.

$\Downarrow$

<u>Thm.</u> Strong MFE with weak control exists and unique in law.

<u>Cor.</u> $A + D + E \Rightarrow$ it's unique strong solution with strong control.

<u>Rmk:</u> Let $\sigma_0 = 0$ or $\sigma = 0$. We can get the degenerate case.