Intro. & Basic Def

(1) First taste of Riemann Geo:

Q1: Given surface Σ and two points $x, y \in \Sigma$. Does it exist a shortest path from x to y? Is it unique?

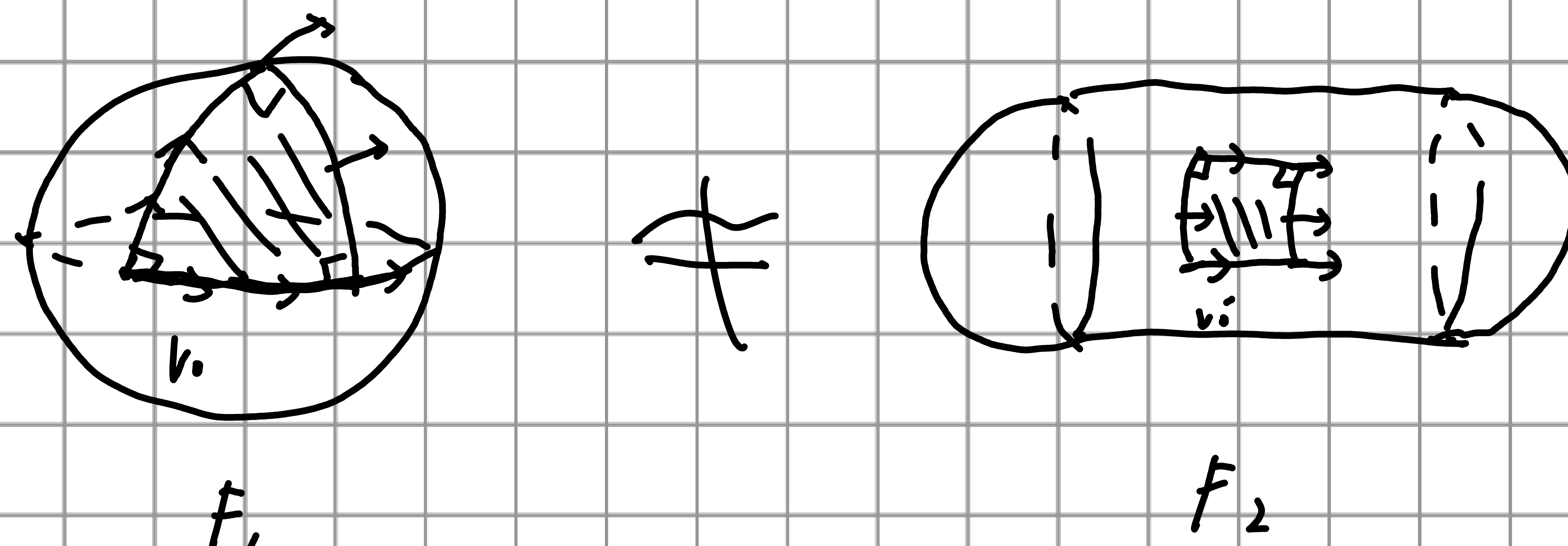
(Distance, Geodesic)

Rmk: If it exists, then we call it a geodesic. It serves a substitute of straight line.

Q2: Given 2 surface Σ, Σ'. Does it exist an isometry from Σ to Σ' (smooth bijection retain length and angle)

(isometry, curvature)

e.g. F_1 and F_2 are not isometric:



If they ← are isom. ⇒ Curvature of them are same!

Since we parallel transport v_0, v_0' around the paths on F_1, F_2 ⇒ F_1 doesn't return the starting v_0 but F_2 does!

Q3: (flatness)

Given surface Σ, is it locally flat (i.e. every point has a nbd isometry to an open set of the flat surface $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$)

e.g. S^2 isn't flat. Since you can't make a sphere flat without breaking or stretching it.

Thm. There's no Riemannian metric on S^2 that's everywhere flat.

Pf: By Gauss-Bonnet Thm.

(2) **Definition:**

Def: i) A n-dim chart (U, x) on set M consists of subset $U \subset M$ and injection $x: U \to \mathbb{R}^n$. St. $x(U) \underset{\text{open}}{\subset} \mathbb{R}^n$

ii) For chart (U, x), (V, y) on M transition map of them is:

$\mathbb{R}^n \supset x(U \cap V) \xrightarrow{y \circ x^{-1}} y(U \cap V) \subset \mathbb{R}^n$

$\mathbb{R}^n \supset y(U \cap V) \xrightarrow{x \circ y^{-1}} x(U \cap V) \subset \mathbb{R}^n$.

iii) The two charts is C^k-compatible if $x(U \cap V)$, $y(U \cap V)$ are open and $y \circ x^{-1}$ and $x \circ y^{-1} \in C^k$.

Rmk: i) (U, x) is interpreted as a local coordinate system.

$x = (x_1 \cdots x_n)$ sometimes is seen as a coordinate. While it can also be seen as a map.

ii) It's important for requiring $x(U \cap V)$, $y(U \cap V)$ are open in Def iii). Since differential can only be treat on open nbd.

iii) $U \cap V$ can be null.

iv) An atlas of class C^k for set M is $\mathcal{A} := \{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ s.t. $M = \bigcup_\alpha U_\alpha$ and they're all C^k-compatible.

Rmk: Almost all atlases are finite.

prop. C^k-compatible is a equi. relation.

Pf: Sym. refl. are trivial. trans. is proved below.

Lemma. (U, x), (V, y) are C^k-compatible on set M. $f: M \to \mathbb{R}^1$. Then $\forall r \leq k$. $f \circ x^{-1} : x(U \cap V) \to \mathbb{R}^1 \in C^r$ $\Leftrightarrow f \circ y^{-1} : y(U \cap V) \to \mathbb{R}^1 \in C^r$

Def: $f \in C^r(M; \mathbb{R}^1)$ if for C^k-atlas A. $k \geq r$. $\forall (U, x) \in A$. $f \circ x^{-1} : x(U) \to \mathbb{R}^1$ is class of C^r.

Next we say atlas A is maximal if it can't be enlarged (May not unique)

Lemma A is C^k-atlas. A' is collection of all charts that're C^k-compatible with all charts in A. Then A' is maximal C^k-atlas that's unique to contain A.

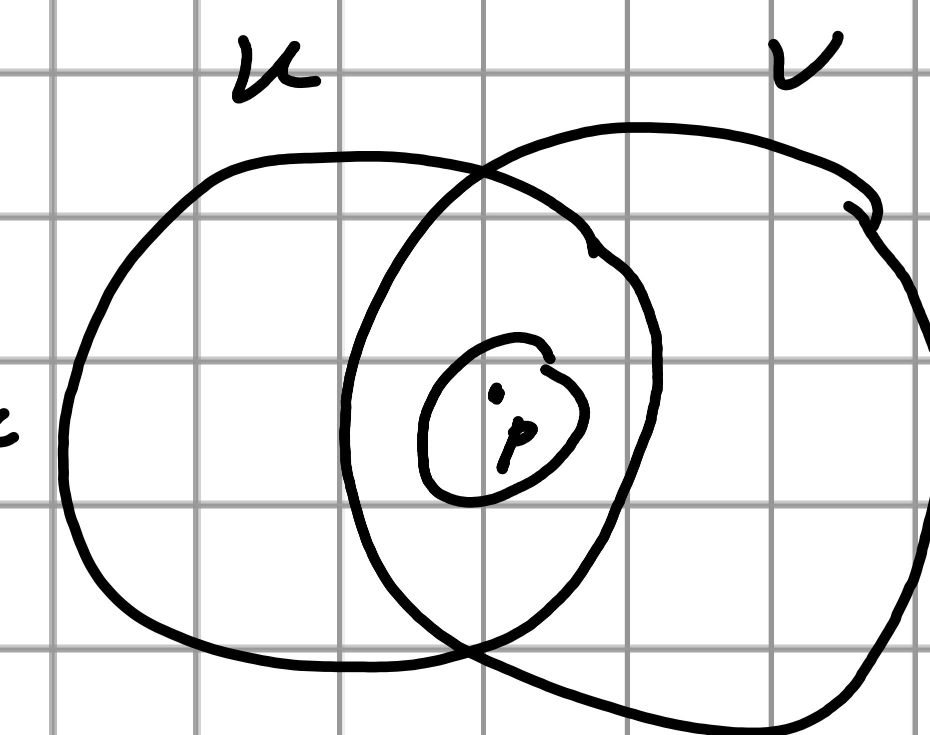
Pf: i) Show: $(\mathcal{U}, x)$ and (V, y) both $\in A$

C^k-comp $\forall (\mathcal{U}_\alpha, x_\alpha)$. $\Rightarrow$ $(\mathcal{U}, x)$

transition property on C^k-compat.

C^k-comp (V, x).

$\forall p \in \mathcal{U} \cap V$. choose



α. st. $p \in \mathcal{U}_\alpha$,

Consider in $x(\mathcal{U} \cap V \cap \mathcal{U}_\alpha)$.

$\mathcal{U}_r \triangleq C^r$-structure

$$\Rightarrow y \circ x^{-1} = (y \circ x_\alpha^{-1}) \circ (x_\alpha \circ x^{-1}) \in C^k$$

Any $\mathcal{U}_r$ can be extended to unique max $\mathcal{U}_s$ struc. $(s < r)$. $\Rightarrow$

$\mathcal{U}_0 \supset \mathcal{U}_1 \cdots \supset \mathcal{U}_\omega$

ii) Uniqueness is obvious by i).

Since $A' \sim A \sim A'' \Rightarrow A' \sim A''$

Cor. Any C^k-chart belongs to unique max atlas.

Def: C^k-Structure on set M is a max atlas of class C^k. on M.

Rmk: By the Lemma above, we can first find the smallest one then extend to max atlas.

prop. For C^0-atlas $A = \{(U_\alpha, x_\alpha)\}_{\alpha\in I}$ on M.

$\Rightarrow$ $\exists$ unique topo on M s.t. [$\forall \alpha$. U_α is open and $x_\alpha : U_\alpha \to x_\alpha(U_\alpha)$ is a homeomorphism.]

Besides. $\forall$ other C^0-compat. chart $(U, x) \notin A$. also satisfies [...].

Pf: i) Uniqueness:

Note $O = \bigcup_{\alpha\in I} (O \cap U_\alpha)$

$O \cap U_\alpha = x_\alpha^{-1}(\underbrace{x_\alpha(O\cap U_\alpha)}_{\subset \mathbb{R}^n})$

The topo structure will be determined by topo on $\mathbb{R}^n$.

2) Existence:

Set $\mathcal{T} = \{ O \subset M \mid x_\alpha(O \cap U_\alpha)$ is open in $\mathbb{R}^n$. $\alpha \in I \}$

Check it's a topo.

3') If $(\mathcal{U}, x) \overset{C^\infty\text{-compat.}}{\sim} (\mathcal{U}_\alpha, x_\alpha)$, $\forall \alpha$.

Show: $O \subset \mathcal{U}$ open in M.

$\iff x(O) \subset \mathbb{R}^n$ open.

By the same argu- in i)

<u>Rmk</u>: To determine the topo. induced by atlas. $\mathcal{A}$. We can just consider on the minimal one $\mathcal{A}' \subset \mathcal{A}$. by the last statement above

<u>e.g.</u> Topo induced by Atlas may be very pathological: Set $M := \mathbb{R}^1 \times \{0,1\} / (t,0) \sim (t,1)$

M is a line with two origins. $t \neq 0$

given a smooth atlas:

$\mathcal{U}_\alpha = \{ [(t,0)] \mid t \in \mathbb{R}^1 \}$. $\mathcal{U}_\beta = \{ [(t,1)] \mid t \in \mathbb{R}^1 \}$

$x_\alpha, x_\beta : \mathcal{U}_\alpha, \mathcal{U}_\beta \to \mathbb{R}^1$. by $[(t,k)] \mapsto t$.

Note $p_j = [(1/j, 0)]$ has 2 limit

points $[(0,0)]$. $[(1,1)]$. !

$\Rightarrow$ It's impossible to def metric.

Next, we will endow M a topo that U_α is open and x_α is homeomorphism

Recall: X is metrizable if it admits a metric s.t. its topo contains all sets that're union of open balls.

Def: Set M is C^k-manifold if it's endowed a C^k-structure and the induced topo is metrizable and separable.

If every chart in atlas has dim $= n$. Then: M is n-dim mfd.

Def: (Alternative)

M is C^k-mfd if C_2. Hausdorff. and locally C^k-diffeo to $\mathbb{R}^m$.

Rmk: i) M can have no dimension if it has more than one components (At most countable)

ii) Disjt. union uncountable many copies of manifolds isn't a manifold since it's not separa.

iii) Actually C^1-manifold can be smooth manifold by removing some charts from its maximal. C^1-atlas.

eg. i) Vector field: $\mathbb{R}^k$ has std. smooth structure from $(\mathbb{R}^k, id)$. But it's possible to define other smooth struc.

ii) Open sets: $O \subset M$. C^k-manifold.

iii) Disjoint union: $\bigsqcup_{j \in J} M_j \triangleq \{(j, t) \mid j \in J. t \in M_j\}$ where $(M_j, \mathcal{A}_j)$ is C^k-mfd.

J need to be countable many to let $\bigsqcup_J M_j$ be a C^k-mfd. and its atlas $\mathcal{A} \triangleq \cup \mathcal{A}_j$.

iv) Dimension zero: discrete topology with metric $d(x,y) = 1 - \mathbb{I}_{\{x=y\}}$.

v) Cartesian product: $M \times N$. with $\mathcal{A} = \{(U_\alpha \times V_\beta, x_\alpha \times y_\beta)\}$. where M, N are C^k-manifolds.

vi) Projective plane $\mathbb{R}P^2 := S^2/_{p \sim -p}$.

vii) Torus: $\mathbb{T}^2 \cong \mathbb{R}^1 \times (\mathbb{R}^1/\mathbb{Z}) / _{(s,[t]) \sim (s+1,[t])}$

Klein bottle $\mathbb{R}^1 \times (\mathbb{R}^1/\mathbb{Z}) / _{(s,[t]) \sim (s+1,[-t])}$

(glue the opposite point on another sides)

The atlas is chosen by viewing it as quotient of $\mathbb{R}^2$, restrict on small nbd st. it contains at most one element in equi. class.