Tensor & Exterior Algebra

(1) <u>Tensor product:</u>

<u>Def</u>: Tensor product of v.s. V and W is a v.s. $V \otimes W$ with natural bilinear map φ

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[dr, "b"'] & V \otimes W \arrow[d, "L"] \\ & X \end{tikzcd}$$

$(v, w) \mapsto v \otimes w$

s.t. $\forall b$. bilinear $b: V \times W \to X$ factors uniquely through $V \otimes W$. i.e. $\exists$ unique L o L. s.t. $L(v \otimes w) = b(v, w)$.

<u>Rmk</u>: i) Elements of form $v \otimes w$ must span $V \otimes W$. Otherwise L won't be unique.

So: $V \otimes W$ has basis $(e_i \otimes f_j)$.

where (e_i), (f_j) is basis of V, W.

ii) $V \otimes W$ is unique, if $\exists X, Y$, $\vee$:

$$\begin{tikzcd} V \times W \arrow[r, "b_X"] \arrow[dr, "b_Y"'] & X \arrow[d, shift right, "L_1"'] \\ & Y \arrow[u, shift right, "L_2"'] \end{tikzcd}$$

$L_1 \circ b_X = b_Y$, $L_2 \circ b_Y = b_X$

So: $L_1 \circ L_2 \circ b_Y = b_Y$

$L_2 \circ L_1 \circ b_X = b_X$

i.e. $L_1 \circ L_2 = id_Y$. $L_2 \circ L_1 = id_X$.

iii) Alternative Definition:

Def: F is free v.s. on set S if:

$$F = \{ \sum_1^n a_i s_i \mid n < \infty,\ a_i \in \mathbb{R},\ s_i \in S \}.$$

Let F is free v.s. over $V \times W$, Let $R \subset F$ spanned by $(v+v', w) - (v,w) - (v',w)$, $(v, w+w') - (v,w) - (v,w')$, $(av, w) - a(v,w)$, $(v, aw) - a(v,w)$. $\Rightarrow F/R = V \otimes W$.

ii) We can define $V \otimes (W \otimes X)$ continually. Since we can prove they're associative under sense of isomorphism. $\Rightarrow$ We can define n tensor product $\bigotimes_1^n V_i$.

Def: A graded algebra is v.s. A. $A = \bigoplus_{k=0}^n A_k$, with associated bilinear multiplication. s.t. $w \cdot \eta \in A_{k+l}$ if $w \in A_k$, $\eta \in A_l$.

We say it communicative if $w \cdot \eta = \eta \cdot w$. / anti-com.. if $w \cdot \eta = (-1)^{kl} \eta \cdot w$.

e.g. $\otimes_* V \triangleq \bigoplus_0^n V^{\otimes k}$. tensor algebra.

It's not communicative or anti-$\square$.

(2) <u>Exterior Algebra</u>:

<u>Def</u>: V is v.s. with dimension $= m$

i) $\Lambda_k V := V^{\otimes k}/S$. $S = \{ \bigotimes_1^k u_i \mid \exists i \neq j \text{ s.t. } u_i = u_j \}$. $v_1 \wedge \cdots v_k \in \Lambda_k V$ is called k-vector. "$\wedge$" is called wedge product.

<u>Rmk</u>: i) Note $(v+w)\wedge(v+w) = 0 \Rightarrow v \wedge w = -v \wedge w$

generally, $v_{\sigma_1} \wedge \cdots \wedge v_{\sigma_k} = \operatorname{sgn}\sigma\,(v_1 \wedge v_2 \cdots \wedge v_k)$

For $\alpha \in \Lambda_k V$, $\beta \in \Lambda_l V$. Then: we have

$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \in \Lambda_{k+l} V$.

ii) $(e_i)_1^m$ is basis of $V \Rightarrow \{ e_{i_1 \cdots i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 \cdots < i_k \leq m \}$ is basis of $\Lambda_k V$. So: $\dim \Lambda_k V = \binom{m}{k}$. $k \leq m$.

$\Rightarrow \dim \Lambda^* V = \sum \binom{m}{k} = 2^m$.

iii) $v^k = \sum_1^m v_i^k e_i \Rightarrow \bigwedge_1^m v^k = \det(v_i^k)\, e_{1\cdots m}$.

iv) $\Lambda_k V$ can also be characterized by:

$$\begin{tikzcd} V^k \arrow[r] \arrow[dr, "\varphi"'] & \Lambda_k V \arrow[d, "L"] \\ & X \end{tikzcd}$$

φ factors uniquely through $\Lambda_k V$.

ii) $v_1 \wedge \cdots \wedge v_k \in \Lambda_k V$ is called simple k-vector

Rmk: i) $\{v_i\}_1^k$ is l.i. $\Leftrightarrow$ $v_1 \wedge \cdots \wedge v_k \neq 0$.

So: Any simple k-form $v_1 \wedge \cdots \wedge v_k \neq 0$ corresp. a k-plane (generated by $\{v_j\}_1^k$)

$v_1 \wedge \cdots v_k$ and $w_1 \wedge \cdots w_k$ corresp. the same oriented k-plane if: they diff. a positive multiple.

ii) The positive multiple is actually ratio of k-areas.

iii) $G_k(V)$ denotes set of oriented k-planes in V. called Grassmannian.

Unoriented Grassmannian is $G_k(V)/\pm$

Rmk Given V inner product and orientation

$\Rightarrow$ Any k-plane has unique orthogonal $(m-k)$-plane. i.e. $G_k(V) \cong G_{m-k}(V)$.

It can extend to linear isometric iso:

$* : \Lambda_k V \to \Lambda_{m-k} V$. $*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_m$.

$\{e_k\}_1^m$ is oriented o.n.b of V.

Similarly $*(e_{i_1}\wedge\cdots e_{i_k}) = e_{j_1}\wedge\cdots e_{j_{m-k}}$.

$i_1 < i_2 \cdots < i_k$. $j_1 < \cdots < j_{m-k}$. $\{j_k\}\cup\{i_k\} = [k]$, $\hat{}$

iv) $\Lambda_* V := \bigoplus \Lambda_k V$ is called exterior algebra. Elements in $\Lambda_* V$ called its multivectors $\Rightarrow$ $\Lambda_* V$ is anticom graded algebra.

Thm. Fix o.n.b. of V. which corresponds $SO(m)$.

We have $G_k(V) \cong SO(m)/(SO(k)\times SO(m-k))$

and $\dim G_k(V) = \binom{m}{2} - \binom{k}{2} - \binom{m-k}{2} = k(m-k)$

Pf: Lemma $SO(n)$ is $n(n-1)/2$ - dim C^∞-mfd.

Pf: $\phi: \mathbb{R}^{n\times n} \simeq \mathrm{Mat}_{n\times n} \xrightarrow{\text{smooth}} \mathrm{Sym}_{n\times n} \simeq \mathbb{R}^{n(n+1)/2}$

$X \mapsto X^T X - I_n$

Note that $SO(n) = \phi^{-1}(\{0\})$

$$D_V \phi^{(X)} = \lim_{h\to 0} \frac{(X+hV)^T(X+hV) - I - X^T X + I}{h}$$

$$= X^T V + V^T X.$$

So $\phi'(X): V \mapsto (X^T V)^T + X^T V$

$\mathrm{Mat}_{n\times n} \to \mathrm{Sym}_{n\times n}$ surjective

$\Rightarrow 0$ is a regular value of ϕ.

So $\dim \phi^{-1}(0) = n^2 - n(n+1)/2$

Note that $SO(n)$ acts transitively on $Gr_k(V)$. (i.e. $\forall M, N \in Gr_k(V)$. $\exists g \in SO(n)$ st. $gM = N$.). And note if H is the stabilizer of some chosen $g \in Gr_k(V)$ (i.e. $\forall h \in H$. $\Rightarrow h \cdot g = g$.). then we have

$SO(n)/H \simeq Gr_k(V)$.

Let g is spanned by $\{e_i\}_1^k \subset \{e_i\}^n$. o.n.b.

$\Rightarrow \forall h \in H$ has form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

Note $H \subset SO(n)$. $\forall h \in H$. $h^T h = I$.

$\Rightarrow h$ has form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Where

$A^T A = I_k$. $B^T B = I_{n-k}$.