Diff. form & Exterior Deri.

(1) Diff. form:

Next, we want to consider dual space of $\Lambda_k V$. Recall $(X/Y)^* \simeq Y^\circ \subset X^*$. Y° is the annihilator of Y. (i.e. $\forall f \in Y^\circ$. $\langle f, y\rangle = 0$. $\forall y \in Y$)

$\Rightarrow \Lambda^k V := (\Lambda_k V)^*$ can be thought as subspace of alternating k-linear map. (i.e. $f(x) = 0$ if $\exists k \neq j$. $x_k = x_j$)

Rmk: $\Lambda^1 V = V^*$.

Def: i) Wedge product on $(\Lambda^k V)$ is define by

$$w \wedge \eta (v_1, \cdots v_{k+l}) := \frac{1}{k!\, l!} \sum_{S_{l+k}} \mathrm{sgn}(\sigma)\, w(v_{\sigma_1} \cdots v_{\sigma_k})$$

$\eta(v_{\sigma(k+1)}, \cdots v_{\sigma(k+l)})$ for $w \in \Lambda^k V$. $\eta \in \Lambda^l V$.

$(w \wedge \eta := w \otimes \eta - \eta \otimes w$. for $w, \eta \in \Lambda^1 V$. $k = l = 1)$

Rmk: i) $(w \wedge \eta) \wedge \gamma = w \wedge (\eta \wedge \gamma)$

ii) $1/k!\,l!$ is chosen for: if (e_i) is basis of V. (w_i) is dual basis of (e_i). $\Rightarrow (w_{i_1} \wedge \cdots w_{i_k})$ is dual basis of $(e_{i_1} \otimes \cdots e_{i_k})$ in $\Lambda^k V$.

iii) $\Lambda^* V = \bigoplus_0^n \Lambda^k V$ with wedge product "$\wedge$" is a anticom. graded algebra of dim $= 2^n$.

iv) $L : V \to W$. induce a pull-back on $\Lambda^k W$: $L^* w(v_1 \cdots v_k) = w(Lv_1 \cdots Lv_k)$.

ii) k-form on M^n is a section on $\Lambda^k T^*M$.

i.e. $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$. which's a module over $\Omega^0(M) = C^\infty(M)$

And let $\Omega^*(M) = \Gamma(\Lambda^* T^*M) = \bigoplus \Omega^k M$.

called exterior algebra on M.

<u>Rmk:</u> i) $f : M \to N$ smooth. We can define pull-back of w: $f^* w$ on M:

$(f^* w)_p (X_1 \cdots X_k) = w_{f(p)}(D_p f X_1 \cdots D_p f X_k)$

And we also have $f^*(w \wedge \eta) = f^* w \wedge f^* \eta$.

(check by definition)

ii) Under local chart (U, φ). We have basis (∂_i), (dx^i) for T_pM, T_p^*M. So $\forall w \in \Omega^k$.

$w|_U = \sum w_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $w_{i_1 \cdots i_k} \in C^\infty(U)$.

(2) Exterior Deri:

Recall for $f \in \Omega^0(M) = C^\infty(M)$. We have $df \in \Omega^1(M) = \Gamma(T^*M)$. Next, we want to generalize d for $\forall$ k-form.

Def: Antiderivation on graded algebra $(\Omega^* M, \wedge)$ is $D: \Omega^* M \xrightarrow{L.O.} \Omega^* M$. s.t. (Leibniz law)

$$D(\omega \wedge \eta) = D\omega \wedge \eta + (-1)^k \omega \wedge D\eta. \quad \omega \in \Omega^k M.$$

Rmk: It's local. (i.e. $\omega = \eta$ on U open $\Rightarrow$ $D\omega = D\eta$ on U).

Pf: By linearity of D. WLOG. $\eta = 0$.

Set $f \in C^\infty(M)$. supp on U. $f(p) = 1$.

$\Rightarrow f\omega \equiv 0$ on M. So:

$$D(f\omega) = 0 = Df \wedge \omega + f \wedge D\omega$$

i.e. $(D\omega)_p = 0$. $\forall p \in U$.

Thm. For mfd M^m. $d: \Omega^0 M \to \Omega^1 M$ has unique $\mathbb{R}$-linear extension to an antiderivation $d: \Omega^* M \to \Omega^* M$. with degree $= 1$. s.t. $d \circ d = 0$

Rmk: We called such d by exterior deri.

Pf: 1) Uniqueness: For $w \in \Lambda^k(M)$, Consider w in local: $w|_U = \sum f_I \, dx^I$. Since $d^2 = 0$, d is $\mathcal{L}_0$. $\Rightarrow d(f_I \, dx^I) \overset{\text{Leibniz}}{=} df_I \wedge dx_I$ $= \sum_i \partial f_I / \partial x_i \, dx_i \wedge dx^I$. is only possibility.

Rmk: We also use d is local here: d only work $\Lambda^k(M)$. So we first use bump func. to extend $w|_U$ to $\tilde{w} \in \Lambda^k(M)$. Note $\tilde{w}|_U = w|_U$ $\Rightarrow D\tilde{w}|_U = Dw|_U$.

And choose different chart (V, ψ). it's still well-def.

2) Next we check d defined in 1) is antideri. :

$d(a\,dx^I \wedge b\,dx^J) = d(ab) \wedge dx^I \wedge dx^J$

$= \cdots = (da \wedge dx^I) \wedge b\,dx^J + (-1)^k \,\square$

3) $d^2 = 0$. is from $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$

Prop. For $f: M^m \to N^n$. We have: $d(f^*\omega) = f^* d\omega$ for $\omega \in \Lambda^* N$.

Pf: Consider locally at $p \in M$. (V, φ) is coordinate of $f(p)$. WLOG. Set $\omega = a\, dy_{i_1} \wedge \cdots dy_{i_k}$.

So: $f^*\omega = f^*a\, df_{i_1} \wedge \cdots df_{i_k}$. where $df_i := f^* dy_i$.

Since $d(f^*a)(X_p) = X_p(f^*a)$

$= f_*(X_p)(a) = da(f_* X_p)$

$= f^*(da)(X_p)$. $\forall X_p \in T_pM$.

$\Rightarrow d(f^*\omega) = f^* d\omega$.

Def: Contraction / interior deri. for $\omega \in \Lambda^k M$ at $X \in \mathfrak{X}(M)$ is i_X. s.t. $i_X\omega(X_2, \cdots X_k) = \omega(X, X_2 \cdots X_k)$. $\in \Lambda^{k-1} M$.

Rmk: For $\omega \in \Lambda^0(M)$. We set $i_X\omega = 0$.

Prop. $\forall X \in \mathfrak{X}(M)$. i_X is antideri. with degree $= -1$. s.t. $i_X^2 = i_X \circ i_X = 0$.