# Integration

(1) Orientations:

Def: i) Orientation on $m$-dim v.s. $V$ is a choice of one of 2 connected components $\Lambda^m V / \{0\}$.

ii) $V$ is oriented by $w$. We call $(e_i)_1^m$ basis is positively oriented if $w(e_1 \cdots e_m) > 0$.

Def: i) Volume form on mfd $M^m$ is a nowhere vanishing $m$-form $w_0 \in \Omega^m M$.

ii) $M$ is orientable if it admits a volume form $w$.

Rmk: i) $W \neq 0$. by conti. $W > 0$ or $W < 0$.

ii) Klein bottle, Mobiüs ... aren't orien.

iii) Std orien. in $\mathbb{R}^m$ is $dx^1 \wedge \cdots \wedge dx^m$.

iv) $S^n$ is orien. : $i : S^n \underset{\text{emb}}{\hookrightarrow} \mathbb{R}^{n+1}$. Assume $n \in NS^n \subset T\mathbb{R}^{n+1}$. $\Rightarrow i^*(\iota_n dx^1 \wedge \cdots dx^{n+1})$ is volume form since $S^n = \mathrm{span}\{x\}^\perp$.

iii) $f : M_1^n \to M_2^n$ is orientation preserved if $\forall$ local chart $(U, x)$ of $M_1$ & $(V, y)$ of $M_2$. we have $D(y \circ f \circ x^{-1})$ has positive determinant.

(v) (Alternative def for orientable)

$M$ is orientable if it admits a coherent oriented atlas $\{(U, \varphi)\}$. i.e. for $(U, \varphi)$ and $(V, \psi)$. $|D(\psi \circ \varphi^{-1})| > 0$.

For $M$ is orientated Riemannian mfd. $\Rightarrow$ $\exists\, \omega \in \Lambda^m TM$. s.t. $\omega_p(e_1 \cdots e_m) = 1$. $\forall p \in M$ where $[e_i]_1^m$ is orientated o.n.b. for $T_pM$. i.e. $\omega = *1$. it gives $M$ Riemannian volume form.

Rmk: i) Consider local chart $(U, \varphi)$ at $p$. $\{\partial_i\}$ is coordinate basis for $T_pM$. Then:

$\partial_i = \sum a_k^i e_k$. $g_{ij} \triangleq \langle \partial_i, \partial_j \rangle = a^i \cdot a^j$.

$\Rightarrow \det(g_{ij}) = (\det(a_{ij}))^2$.

Assume $\varphi$ is oriented. So: $\det(a_{ij}) > 0$.

$\Rightarrow \omega_p(\partial_1, \cdots \partial_m) = \det(a_{ij})\, \omega_p(e_1 \cdots e_m)$

$= \det(g_{ij})^{\frac{1}{2}}$.

So: $\Omega_p = (\det g)^{\frac{1}{2}} dx_1 \wedge \cdots dx_m$, where $dx_i(\partial_j) = \delta_{ij}$.

iv) $\forall$ $m$-volume form $W$ in $(M, g)$. $W = f\Omega = *f$. for some $f \in C^\infty(M)$.

(2) <u>Integration:</u>

Recall a bdd func. $f: D \subseteq_{\text{bdd}} \mathbb{R}^m \to \mathbb{R}^1$ is Riemannian integrable $\iff$ $\bar{f}$ is conti. a.e. where $\bar{f} = f$ on $D$. $\bar{f} = 0$ on $D^c$.

So: $I_D$ is Riemannian integrable $\iff$ $\partial D$ has null measure. & $D$ is bdd

<u>Def</u>: $W \in \Omega_c^m(U) \triangleq$ {$m$-form with cpt supp}.

$U \subseteq \mathbb{R}^m$. write $W = f\, dx_1 \wedge \cdots \wedge dx_m$.

Set $\int_U W \triangleq \int_U f\, dx_1 dx_2 \cdots dx_m$.

<u>Lemma</u>. $\varphi: U \to V$. diffeo. of connected open sets $\subseteq \mathbb{R}^m$. $W \in \Omega_c^m(V)$. Then:

$\int_U \varphi^* W = C_\varphi \int_V W$. $C_\varphi = \pm 1$ depd on $\varphi$.

**Pf**: Consider in local chart of $U$ with coordinate $(x^i)$ and $(y^i)$ of $V$.

$\Rightarrow J = (\partial(y^j \circ \varphi)/\partial x^i)$. $w = f\, dy_1 \wedge \cdots dy_m$.

So $LHS = \int_U \varphi^* f \quad \varphi^* dy_1 \wedge \cdots \wedge \varphi^* dy_m$

$= \int_U (\det J) f \circ \varphi \, dx_1 \wedge \cdots dx_m$.

Next, we define integration on oriented mfd $M^m$. if $w \in \Omega_c^m(M)$.

For local chart $(U, \varphi) \subset M$. We can set $\int_U w := \int_{\varphi(U)} (\varphi^{-1})^* w$.

**Rmk**: It's well-def. If $\psi$ is another chart, then:

$$\int_{\varphi(U)} (\varphi^{-1})^* w = \int_{\psi(U)} (\varphi \circ \psi^{-1})^* (\varphi^{-1})^* w = \int_{\psi(U)} (\psi^{-1})^* w.$$

$\Rightarrow$ We can find PoU $\{f_\alpha\}$ subordinate to orientated atlas $\{(U_\alpha, \varphi_\alpha)\}$. And $w = \Sigma f_\alpha w$.

So: $\int_M w := \sum_\alpha \int_{U_\alpha} f_\alpha w$. $(\int_{-M} w \overset{\Delta}{=} -\int_M w)$

**Rmk**: It's well-def. if $\{(V_\beta, \psi_\beta)\}$ is another atlas. with PoU $\{g_\beta\}$ then:

$$\sum_\alpha \int_{U_\alpha} \sum f_\alpha w = \sum_\alpha \int_{U_\alpha} f_\alpha \sum_\beta g_\beta w$$

$$= \sum_{\alpha,\beta} \int_{U_\alpha} f_\alpha g_\beta w = \sum_\beta \int_{U_\beta} g_\beta w .$$

Rmk: i) For $m=0$. $M = \sum p_i - \sum q_j$ (it have orien.)

$\Rightarrow \int_M f = \sum f(p_i) - \sum f(q_j)$.

ii) On orientated Riemannian mfd $M$. We define $\int_M f \, d\mathrm{Vol} := \int_M f \nu = \int_M * f$ for $f \in C_c^\infty(M)$. (change orien. on $-M$. then volume will also be $-\nu$. $\Rightarrow \int_{-M} f d\mathrm{Vol} = \int_M f d\mathrm{Vol}$)

iii) $\mathrm{Vol}(D) := \int_M I_D \, d\mathrm{Vol}$ for $D \subset M$ in ii).