Connections

(1) Motivation on $\mathbb{R}^n$:

- Between $X, Y \in \mathfrak{X}(M)$. We have Lie deri. on them, but it's not satisfied since we can only deri. it along the flow.
- Next, we will introduce connection, which is kind of "directional derivative."
- Consider $X = X^i \partial_i$, $Y = Y^i \partial_i \in \mathfrak{X}(\mathbb{R}^m)$. We have $D_X Y|_x = \lim_{t \to 0} Y(x+tX) - Y(x) / t = X^i \partial_i (Y_j) \partial_j$.

i.e. directional deri. of Y along v.f. X.

Satisfy: i) $D_{fX} Y = f D_X Y$ ii) $D_X(fY) = (Xf) Y + f D_X Y$.

<u>Rmk</u>: Lie deri. on general mfd M doesn't satisfy i) above! So it's not kind of directional deri.

(So, it's not a connection)

If we use the def as in $\mathbb{R}^m$ above.

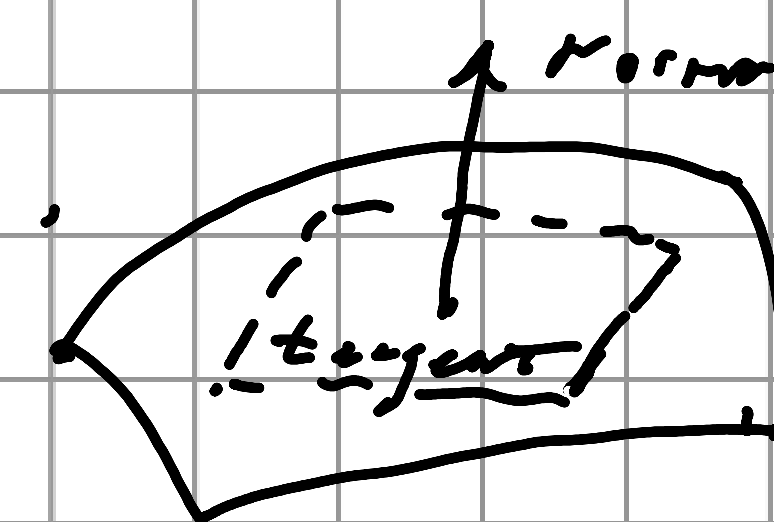
i) $x+tX$ isn't pt in M. We can replace it by $\gamma(t)$. s.t. $\gamma(0)=x$. $\gamma'(t)=X_{\gamma(t)}$.

ii) Consider in i): $Y(\gamma(t))$ & $Y(x)$ belong to 2 diff. v.s. $\Rightarrow$ Difference doesn't make sense.

• We can also consider to embed M into $\mathbb{R}^n$. Consider $M^m \subset \mathbb{R}^n$. submfd.

For $X: M \to T\mathbb{R}^n \cong \mathbb{R}^n$. & $\tilde{X}: M \to \mathbb{R}^n$.

Note $T_p\mathbb{R}^n = T_pM \oplus N_pM$. where $N_pM = (T_pM)^\perp$ is v.s. normal to M.



Def. $Z^{\parallel}$ & $Z^{\perp}$ is orthogonal proj. on T_pM. & N_pM resp.

For $\gamma: [a,b] \to M$. We set $d^{\gamma}X/dt := d(\tilde{X}\circ\gamma)/dt$, deri. of X along γ.

Def: $X \in \mathfrak{X}(M)$. $M \subset \mathbb{R}^n$. γ is curve in M. $D^{\gamma}X/dt := Z^{\parallel}(d^{\gamma}X/dt)$ is called covariant derivative (i.e. v.f along γ and tangent to M).

<u>Rmk</u>: Deri. of tangent v.f. may produce both tangent & normal component.

<u>e.g.</u> $S^2 \subset \mathbb{R}^3$. $\gamma(t) := (\cos t, \sin t, 0)$. Def $X_{\gamma_t} := \gamma_t' \Rightarrow D^2 X/dt = Z''(dX_{\gamma t}/dt)$

$$= Z''(\gamma_t''). = 0.$$

Since $\gamma_t'' = -\gamma_t$ is normal to S^2.

<u>Rmk</u>: We define such curve $(Z''(\gamma_t'') = 0)$ is geodesic for $M \underset{\text{embd}}{\hookrightarrow} \mathbb{R}^n$.

Next, we find coordinate expressions for covariant derivative:

For (u, φ) local chart for M. $V = \varphi(u) \subset \mathbb{R}^m$. Assume $\{u^i\}$. $\{x^\alpha\}$ are coordinate for $\mathbb{R}^m$ & $\mathbb{R}^n$. Note: $(x^1, \cdots x^n) \circ \varphi^{-1} \stackrel{\Delta}{=} \psi = (\psi^1, \cdots \psi^n) : V \subset \mathbb{R}^m \xrightarrow{\varphi^{-1}} U \subset M \xrightarrow{x} \mathbb{R}^m$

$\int_0 : \partial_i = \psi_*(\frac{\partial}{\partial u^i}) = \sum_\alpha^n \frac{\partial \psi^\alpha}{\partial u^i} \frac{\partial}{\partial x^\alpha}$. St. $(\partial_i)_1^m$ is coordinate frame in TU.

Express $Y_{\gamma(t)} = \sum_1^m b^i(t) \partial_i$. v.f. Y along γ_t.

$$\Rightarrow \begin{cases} d^{\varphi} Y/dt = \sum_{i}^{m} \frac{db^i}{dt}\partial_i + b^i\, d\partial_i/dt. \\ D^{\varphi} Y/dt = \sum_{i}^{m} \frac{db^i}{dt}\partial_i + b^i\, \pi''(d\partial_i/dt). \end{cases}$$

Since $(\partial_i) \subset TU = TM$. $\Rightarrow \pi''(\partial_i) = \partial_i$.

Note $\pi''(d\partial_i/dt) = \pi''(\frac{d}{dt} \sum_{i}^{n} \frac{\partial \varphi^{\alpha}}{\partial u^i} \frac{\partial}{\partial x^{\alpha}})$

$$= \sum_{\alpha} \sum_{j} \frac{\partial^2 \varphi^{\alpha}}{\partial u^i \partial u^j} \cdot \frac{du^j}{dt}\, \pi''(\frac{\partial}{\partial x^{\alpha}}).$$

Assume $(C_{\alpha}^{k})_k \subset C^{\infty}(U)$. $\pi''(\frac{\partial}{\partial x^{\alpha}}) = \sum_{k}^{m} C_{\alpha}^{k}\, \partial_k$.

<u>Def</u>: Christoffel symbols $\Gamma_{ij}^{k} := \sum_{\alpha} \frac{\partial^2 \varphi^{\alpha}}{\partial u^i \partial u^j} C_{\alpha}^{k}$.

<u>Rmk</u>: In $\mathbb{R}^m$. $\Gamma_{ij}^{k} = \Gamma_{ji}^{k} \in C^{\infty}(U)$.

So: $D^{\varphi} Y/dt = \sum_{k}^{m} (\frac{db^k}{dt}\partial_k + \sum_{ij} \Gamma_{ij}^{k}\, b^i \frac{du^j}{dt}\partial_k)$

<u>Rmk</u>: For $Y_0 = \sum b_0^k \partial_k \in \mathfrak{X}(M)$. $D^{\varphi} Y/dt\,|_{p=\gamma(t_0)}$

doesn't depend on the whole Y.

rather only its velocity vector

$X_p = \gamma'(t_0)$ (since $db^k/dt\,|_{\gamma(t_0)=p} =$

$d\, b^k \circ \gamma(t)/dt\, |_{\gamma(t_0)=p}$)

$\Rightarrow$ Note that set $X_p = \sum a^i \partial_i$. so $a^i = \frac{du^i}{dt}$

Then: $\alpha(b^k(\gamma(t))/dt|_{t_0} = \sum_j a^j(\partial_j b^k) = X_p(b^k)$

So. for general case. i.e. $\forall (a^i). (b^i)$

for $X = \sum a^i \partial_i$. $Y = \sum b^i \partial_i \in \mathfrak{X}(M)$.

$$\nabla_{X_p} Y := \sum_k (X_p(b^k) + \sum_{ij} \Gamma_{ij}^k b^i a^j) \partial_k$$

$$= \sum_{k,j} (a^j(\partial_j b^k) + \sum_i \Gamma_{ij}^k b^i a^j) \partial_k.$$

$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), (X, Y) \mapsto \nabla_X Y$.

is called connection for $M \subset \mathbb{R}^n$.

<u>Rmk</u>: i) It's bilinear & satisfy property i)

and ii) mentioned at the beginning

$\Rightarrow \nabla$ is truly directional deri.

ii) Actually ∇ above is LC connection!

iii) It's consistent with covar deri. as constructed

(2) <u>Connections</u>:

Next, we consider connection operates on section $\Gamma(E)$. where $E \to M^m$ is v.b.

<u>Def</u>: Connection on E is a bilinear map:

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), (X, \sigma) \mapsto \nabla_X \sigma.$$

Satisfy: i) $\nabla_{fX}\sigma = f\nabla_X\sigma$ (tensoriality)

ii) $\nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X\sigma$. (product rule)

for $\forall f \in C^\infty(M)$.

Rmk: i) Nontrivial connections always exist.: We can construct local connection as in (1). And use PoU to glue them up

ii) Set of all connections doesn't form a linear space. (but a convex set)

iii) Tensoriality $\Rightarrow$ $(\nabla_X\sigma)_p$ will only depend on X_p: (So write $\nabla_X\sigma(p) = \nabla_{X_p}\sigma$)

$X = \sum X_i\partial_i$. locally expression.

$\Rightarrow (\nabla_X\sigma)_p = \sum X_i(p)(\nabla_{\partial_i}\sigma)_p$

But $\sigma(p) = \tilde{\sigma}(p) \not\Rightarrow (\nabla_X\sigma)_p = (\nabla_X\tilde{\sigma})_p$

iv) We can see connection as:

$\nabla_\cdot\sigma : TM \to E$. $X \mapsto \nabla_X\sigma$.

i.e. $\nabla_\cdot\sigma \in L(TM, E) = E\otimes T^*M$.

$\Rightarrow \nabla : \sigma \in \Gamma(E) \mapsto \Gamma(E\otimes T^*M)$.

v) U is local chart. of M. consider

$\{\partial_i\}_1^m$ is coordinate frame for TM.

$\{e_j\}_1^{\mathrm{rank}E}$ is frame for E.

Express $\nabla_{\partial_i} e_j = \sum_k \Gamma_{ij}^k e_k \in E$ locally

We call Γ_{ij}^k is Christoffel symbols

corresp'd to ∇ in this case.

So, for $X = \sum v_i \partial_i$, $\sigma = \sum \sigma^j e_j$.

$$\nabla_X \sigma = \sum_{i,j} v_i (\partial_i \sigma^j + \sum_k \Gamma_{ik}^j \sigma^k) e_j$$

follows from prop. i). ii) & bilinearity.

which's consistent with $\mathbb{R}^n$ - case!

vi) We can view a connection as complement of some horizontal subspace $\hat{=} \{ D_p \sigma(X_p)$: $\nabla_{X_p} \sigma = 0$ for section $\sigma : M \to E.\}$. i.e. definition from parallel transport.

prop. (locality)

i) $\forall\, U \underset{\mathrm{open}}{\subseteq} M$. $\sigma|_U = \tilde{\sigma}|_U$, $X|_U = \tilde{X}|_U \Rightarrow$

$\nabla_X \sigma = \nabla_{\tilde{X}} \tilde{\sigma}$ on U.

ii) $X(p) = \tilde{X}(p)$. $\Rightarrow \nabla_X \sigma(p) = \nabla_{\tilde{X}} \sigma(p)$.

iii) (local at least along a curve)

$\gamma : (-\varepsilon, \varepsilon) \to M$ smooth curve. s.t.

$\gamma(0) = p$. $\gamma'(0) = v$. $X \in \mathfrak{X}(M)$. $\sigma, \tilde{\sigma}$

$\in \Gamma(E)$. s.t. $\sigma(\gamma(t)) = \tilde{\sigma}(\gamma(t))$. $\forall$

$t \in (-\varepsilon, \varepsilon) \Rightarrow \nabla_X \sigma(p) = \nabla_X \tilde{\sigma}(p)$.

Pf: ii) is from Rmk iii) above

i) Assume $X = 0$ or $\sigma = 0$ on U.

and show: $\nabla_X \sigma \; (= \nabla_{\tilde{X}} \sigma = \nabla_{\tilde{X}} \tilde{\sigma}) = 0$.

Choose $f \in C_0^\infty(U)$. s.t. $f \equiv 1$ on $\overset{\text{nbhd of } p}{V} \subset U$

$\Rightarrow X = f \cdot X$. $\sigma = f \cdot \sigma$ on U.

Use product rule & bilinearity.

iii) Assume $\sigma = 0$ along γ. And prove:

$\nabla_v \sigma(p) = 0$.

Consider $\delta < \varepsilon$. locally assume:

$x_2(t) = \cdots x_m(t) = 0 \quad \forall t \in (-\delta, \delta)$.

$\Rightarrow v = a \partial_1$. And $\sigma = \sum \sigma^j e_j$

satisfies $\sigma^j(x_1, 0 \cdots 0) = 0$. on nbhd of p.

So: $\sigma^j(p) = 0 \Rightarrow \partial_1 \sigma^j(p) = 0$.

(3) Levi-Civita Connection:

Def: (M, g) is Rieman mfd. ∇ is a connection on $E = TM$.

1) ∇ is torsion-free if $\nabla_X Y - \nabla_Y X = [X, Y]$. $\forall X, Y \in \mathfrak{X}(M)$.

Rmk: $\Rightarrow$ It's too hard to hope $\nabla_X Y = \nabla_Y X$.

So we weaken it.

ii) ∇ is torsion-free $\Leftrightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$. under cordinate basis.

ii) ∇ is compatible with g if:

$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Rmk: Also say g is parallel with ∇.

iii) ∇ is called Levi-Civita connection on (M, g) if ∇ is torsion-free and compatible with g.

Thm. $\forall$ Rieman mfd (M, g) has a unique Levi-Civita connection, characterized by:

$2g(\nabla_X Y, Z) = X g(Y,Z) + Y g(X,Z) -$
$Z g(X,Y) + g([X,Y],Z) - g([X,Z],Y)$
$- g([Y,Z],X)$. ω

Rmk: So in $\mathbb{R}^m$ - case, we can construct a connection only depends on g rather embedding

Pf: 1) Uniqueness:

check any Levi-Civita connection satisfies the formula:

Replace $(X g(Y,Z))$ & $[Y,Z]$ by property of ∇

2) Show the formula define a Levi-Civita connection:

Set RHS $= \omega(Z)$. check it's tensorial: $\omega(fZ) = f\omega(Z)$ for $\forall f \in C^\infty(M)$. by using formula for $[X, fZ]$ & Leibniz rule

W is also linear $\Rightarrow W \in \Omega^1(M)$.

<u>Lemma.</u> $\forall\, \omega \in \Omega^1(M)$. $\exists\, W \in \mathfrak{X}(M)$. s.t.

$g(W, Z) = \omega(Z)$. $\forall\, Z \in \mathfrak{X}(M)$.

<u>Pf</u>: Consider in coordinate $(U_\alpha, \varphi_\alpha)$.

$\omega = \sum w^i dx^i$. $Z = \sum z^i \partial_i$.

$W_\alpha = \sum \widetilde{w}^i \partial^i$. s.t. $dx^i(\partial_j) = \delta_{ij}$.

$\Rightarrow \sum \widetilde{w}^i z^j g_{ij} = \sum w^i z^i$

$\Rightarrow \sum_i^m z^i (w^i - \sum_j^m g_{ij} \widetilde{w}^j) = 0$.

g is nondegenerate $\Rightarrow$ solve $\widetilde{w}^i$.

Let (f_α) is P.o.U sub to (U_α).

$\Rightarrow \sum f_\alpha W_\alpha = W$.

So. $\exists\, W \in \mathfrak{X}(M)$. s.t. $2g(W, Z) = \omega(Z)$.

Now $(X, Y) \mapsto W$. is bilinear. And check it satisfies tensorial and product rule. So $(X, Y) \mapsto W$ is conn.

3') Check the connection is Levi-Civita by

$g(X, Z) = 0$ for $\forall Z \Rightarrow X \equiv 0$.

Rmk: We locally express Levi-Civita connection in coordinate U:

$$\partial_k g_{ij} = g(\nabla_{\partial_k}\partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_k}\partial_j)$$

$$= \sum_l \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il}.$$

$$\overset{\Delta}{=} \Gamma_{kij} + \Gamma_{kji}.$$

By sym: $\Gamma_{ij}^k = \Gamma_{ji}^k \Rightarrow \Gamma_{ijk} = \Gamma_{jik}$.

So: $2\Gamma_{ijk} = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}$.

$\Rightarrow \Gamma_{ij}^k = \sum_l (g^{-1})_{kl} \Gamma_{ijl} = f(g)$.

So the connection $\Leftrightarrow$ Christoffel symbols only depend on (g_{ij}).