# Parallel Transport

(1) Definition:

Def: For connection $\nabla$ on $M \to E$.

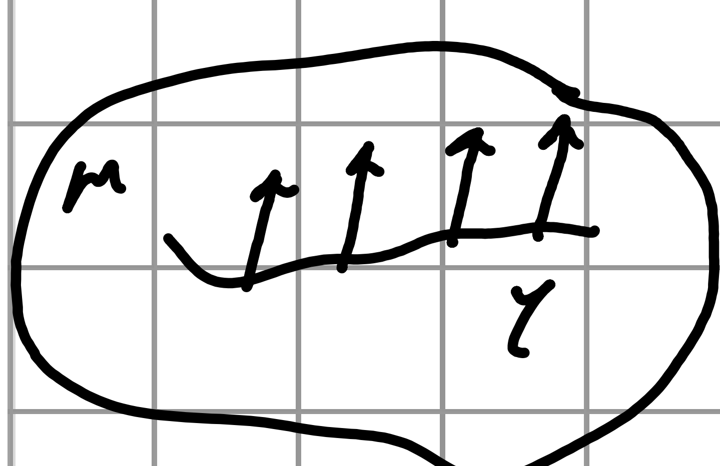
$\sigma \in \Gamma(E)$ is parallel along $\gamma : I \to M$ smooth if covariant deri. along $\gamma$

$$\frac{D^\gamma \sigma}{dt} := \nabla_{\gamma'}\sigma = 0.$$

e.g. $M = \mathbb{R}^n$. $E = TM$. $\nabla_X Y := X(Y^j)\partial_j$ is LC connection with $\Gamma_{ij}^k \equiv 0$. For $Y = Y^j \partial_j$

$$\Rightarrow \quad 0 = \nabla_{\gamma'(t)} X = \gamma'(t)(X^i)\partial_i$$

$$= \frac{d}{dt}(X^i \circ \gamma)\partial_i$$




i.e. $X^i \equiv$ const along $\gamma$ $\Leftrightarrow$ $X$ is const vector field along $\gamma$.

Thm: $\forall \gamma : [a,b] \to M$. $\forall t_0 \in [a,b]$. $\forall X_0 \in E_{\gamma(t_0)}$

$\Rightarrow \exists$ unique section $X$ parallel along $\gamma$

st. $X_{\gamma(t_0)} = X_0$

Pf: WLOG. Consider locally under $\varphi$ :

Assume $X_0 = X_0^j \partial_j|_{\gamma(t_0)}$. $X = X^j \partial_j$.

$0 = \nabla_{\dot\gamma(t)} X \overset{\text{(p.p. i), ii))}}{=} dX^j(\gamma(t))/dt \, \partial_j + X^j(\gamma(t)) \nabla_{\dot\gamma(t)} \partial_j$.

Set $\nabla_{\dot\gamma(t)} \partial_j = \sum a_{kj}(t) \partial_k$.

$\Rightarrow$ We get a linear ODE : $1 \le j \le m$

$f_j'(t) + \sum f_k(t) a_{kj}(t) = 0$, $f_k(t) = X^k(\gamma(t))$.

Apply basic ODE Thm on $(f_1 \cdots f_m)^T$.

Def: $\nabla$ is connection on $E \to M$. $\gamma : I \to M$. smooth curve. $P_{t_0}^t(\gamma) : E_{\gamma(t_0)} \to E_{\gamma(t)}$ is parallel transport define by:

$P_{t_0}^t(\gamma) : \sigma_{\gamma(t_0)} \mapsto \sigma_{\gamma(t)}$. where $\sigma_{\gamma(t)}$ is the ODE solution to datum $\sigma_{\gamma(t_0)}$.

Rmk: $\gamma$ can be piecewise smooth.

Lemma. $P_{t_0}^t(\gamma)$ is linear isomorphism.

Pf: i) linearity : $X_t^{x_0}$, $X_t^{\tilde{x}_0}$ are solution to ODE with datum $x_0$. $\tilde{x}_0$ $\Rightarrow$

$\lambda X_t^{x_0} + X_t^{\tilde{x}_0}$ is solution to ODE with datum $\lambda X_0 + \tilde{Y}_0$.

2) isomorphism: Let $-\gamma(s) = \gamma(a+b-s)$

$$\Rightarrow P_{t_0}^{t}(\gamma) \circ P_{a+b-t}^{a+b-t_0}(-\gamma) = id_{E_{\gamma(t_0)}}$$

<u>prop.</u> (characterization of connection)

$\nabla$ is connection on $M \to TM$. If $\gamma: [a,b] \to M$ smooth curve. St. $\gamma(a) = p$

$\gamma'(a) = X_0 \in T_pM$. Then: $\forall Y \in \mathfrak{X}(M)$. we have:

$$\nabla_{X_0} Y(p) = \lim_{t \to t_0} \frac{P_{t_0}^{t}(\gamma)^{-1} Y_{\gamma(t)} - Y_{\gamma(t_0)}}{t - t_0}$$

<u>Rmk:</u> Connection can be used to define parallel transp. This is a converse statement.

<u>Pf:</u> $\{e_i\}_1^m$ is basis of $T_pM$. $\Rightarrow$

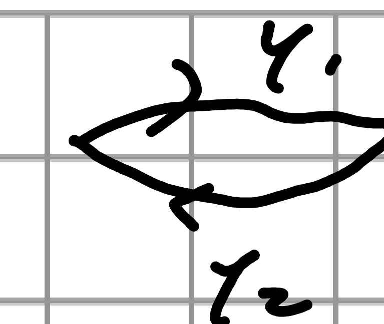
$\{e_i(t) := P_{t_0}^{t}(\gamma)(e_i)\}$ is basis of $T_{\gamma(t)}M$. So: $Y_{\gamma(t)} = Y_i(\gamma(t)) e_i(t)$

$RHS = (Y_i(\gamma(t)))' e_i$

$LHS = \nabla_{X_0} Y_i(p) e_i + \nabla_{X_0} e_i(t) Y^i(p)$

$= (Y_i(\gamma(t)))' e_i = RHS.$

<u>Def:</u> $Hol_p(\nabla) := \{P_0^1(\gamma) \mid \gamma \text{ is loop based at } p\}$

Rmk: i) Consider $P_0^1(\gamma_1) \circ P_1^0(\gamma_2)^{-1} \Rightarrow$ 


$Hol_p(\nabla)$ has nontrivial element.

ii) $Hol_p(\nabla) < GL(E_p)$. called Holonomy group. actually it's a Lie subgroup.

Next, we consider $\nabla$ is on $M \to TM$. where $(M, g)$ is a Riemann mfd.

Prop. For $\nabla$ is compatible with $g$. Let $X, Y \in \mathfrak{X}(M)$ both parallel with $\gamma$. $\Rightarrow$ $g(X, Y)$ is const along $\gamma$.

Pf: $\frac{d}{dt} g_{\gamma(t)}(X_{\gamma(t)}, Y_{\gamma(t)}) = \gamma_t' \, g(X, Y)$

$= g(\nabla_{\gamma_t'} X, Y) + g(X, \nabla_{\gamma_t'} Y) = 0$

Rmk: i) It means parallel vector field has const. length. (let $Y = X$). & has const angle to each others.

$\Rightarrow$ Parallel transp. is orth. map

So: $Hol_p(\nabla) < O(T_pM)$ for metric connection $\nabla$.

iii) $\forall$ metric connections $\nabla$, $\tilde{\nabla}$ on $(M, g)$. Their parallel transp. along $\forall \gamma$ only differs a rotation.

<u>Cor.</u> $\nabla$ is compatible with $g \iff \forall \gamma$ $\forall$ parallel v.f $X, Y$ along $\gamma$. We have $g(X, Y) \equiv$ const along $\gamma$.

<u>Pf.</u> ($\Leftarrow$) take $\gamma$ is flow of $Z \in \mathfrak{X}(M)$. $\gamma(t_0) = p$. $\forall p$. $\forall Z_p$.

<u>Prop.</u> $\nabla$ on $M$ is compatible with $g \iff \forall P_{t_0}^{t}(\gamma)$ is isometry : $T_{\gamma(t_0)} M \to T_{\gamma(t)} M$.

<u>Pf:</u> $\{P_{t_0}^{t}(\gamma)(e_i)\}$ parallel along $\gamma$. It's cor. of above

(2) <u>Geodesic:</u>

Fix $\nabla$ is Levi-Civita connection in $(M, g)$.

<u>Def.</u> $\gamma: [a, b] \to M$ smooth curve is geodesic if $\nabla_{\gamma'(t)} \gamma'(t) = 0$.

<u>Rmk:</u> i) $\gamma$ has const. speed.

ii) It depends on parametrization:

Set $\tilde{\gamma}(s) = \gamma(t(s))$. Then:

$\nabla_{\tilde{\gamma}(s)'} \tilde{\gamma}(s)' = \nabla_{\square} (\gamma'(t(s))\, t'(s))$

$\overset{\text{lin.}}{=} t''(s)\gamma'(t(s)) + t'(s)^2 \nabla_{\gamma'(t(s))} \gamma'(t(s))$

$= t''(s)\, \gamma'(t(s))$

(Note that for scalar func. $f: \mathbb{R}^1 \to \mathbb{R}^1$, $\nabla_X f = L_X f = \langle f(x) = f'(t)$ for $\forall X$, vector field)

So: $\tilde{\gamma}(s)$ is geodesic if $t(s) = as+b$ for $a, b \in \mathbb{R}^1$.

<u>prop.</u> $\varphi: (M, g) \to (N, h)$ is a isometry map. (i.e. homeo. preserves Rieman. metric) Then: $\gamma$ is geodesic in $M \Rightarrow \varphi(\gamma)$ is geodesic.

<u>Pf:</u> <u>Lemma.</u> Levi-Civita connection is preserved in isometry $\varphi$.

<u>Pf:</u> Def $\tilde{\nabla}_{X'} Y' = \varphi_*(\nabla_X Y)$ where $X' = \varphi_*(X)$, $Y' = \varphi_*(Y)$, i.e. $\tilde{\nabla}_X Y = (\varphi^{-1})_* \nabla_{(\varphi^{-1})_* X} (\varphi^{-1})_* Y$. check it's connection and LC prop.

(Or directly check it satisfys Koszul)

$\Rightarrow$ It follows from uniqueness of LC.

<u>Rmk</u>: i) $\varphi$ also preserves induced metrics.

ii) If $h = (\varphi^{-1})^* g$. $\Rightarrow$ $\forall$ homeo. $\varphi$ is isometry

<u>Thm</u>: $\forall p \in M$. $X_p \in T_pM$. $\Rightarrow \exists\ \varepsilon > 0$. and unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$. s.t. $\gamma(0) = p$ & $\gamma'(0) = X_p$. Besides, $\gamma(t. p. X_p)$ depends on $p \cdot X_p$. smoothly.

<u>Pf</u>: As discussed above, choose $(U, \varphi)$ local chart. $\tilde{\gamma}(t) \stackrel{\Delta}{=} \varphi(\gamma(t)) = (X_t^i) \subset \mathbb{R}^m$

So: $\tilde{\gamma}'(t) = x^i(t)' \partial_i$.

$\nabla_{\tilde{\gamma}'(t)} \tilde{\gamma}'(t) = \tilde{\gamma}'(t)(x^i(t)')\partial_i +$
$x^i(t)' x^j(t)' \nabla_{\partial_i} \partial_j$.

$\nabla_{\partial_i} \partial_j = \sum \Gamma_{ij}^k \partial_k$. Then we have

ODE $\stackrel{(*)}{:}$ $X^k(t)'' + x^i(t)' x^j(t)' \Gamma_{ij}^k = 0$. $1 \le k \le m$.

<u>Rmk</u>: The ODE is nonlinear. So we can't hope it exists on the whole $\mathbb{R}^1$.

<u>Def</u>: $(M, g)$ is complete if $\forall$ geodesic $\gamma$ can be extended to $\mathbb{R}^1$

**Thm** (Hopf–Rinow)

$(M, g)$ is geodesically complete $\Leftrightarrow$ $M$ is complete as metric space.

**Rmk**: Any cpt mfd is complete.

**Thm**. (Determine a geodesic)

$\gamma : [a,b] \to M$. smooth curve. If $\exists\ \varphi$ iso-metry $: M \to M$. s.t. fixed pt set of $\varphi = \gamma[a,b]$. Then: $\gamma$ is normal geodesic (i.e. $|\gamma'(t)| = 1$).

**Pf**: If $\beta(t)$ is geodesic. s.t. $\beta(t_0) = \gamma(t_0)$, $\beta'(t_0) = \gamma'(t_0)$. for $t_0 \in [a,b]$.

$\Rightarrow \varphi(\beta(t))$ is also geodesic. with same initial value. $\overset{\text{local}}{\underset{\text{uni.}}{\Rightarrow}} \varphi(\beta(t)) = \beta(t)$.

So: $(\beta(t))_{[a,b]}$ is part of $\gamma[a,b]$.

(3) Exponential maps:

**Def**: $\mathcal{D}\Sigma := \{ (p, X_p) \mid \gamma(t; p, X_p)$ defined on the interval $\supseteq [0,1]$ is geodesic $\}$.

**Rmk**: i) $\Sigma = TM \Leftrightarrow (M, g)$ is complete

ii) Note $\tilde{\gamma}(t) = \gamma(\lambda t; p, X_p)$ is geodesic with $\tilde{\gamma}(0) = p$. $\dot{\tilde{\gamma}}(0) = \lambda X_p$.

So: $\tilde{\gamma}(t) = \gamma(t; p, \lambda X_p)$. by uniqueness. We have: if $(p, X_p) \in \Sigma$. $\Rightarrow \exists \varepsilon > 0$. s.t. $(p, \varepsilon X_p) \in \Sigma$.

iii) Exponential map $\exp: \Sigma \to M$ is def by $(p, X_p) \mapsto \exp_p(X_p) := \gamma(1, p, X_p)$.

eg. $(\mathbb{R}^n, g_0)$. $\exp_p(X_p) = p + X_p$.

Rmk: $\exp$ is smoothly depend on $(p, X_p)$.

Lemma. $\forall p \in M$. If we see $T_0(T_pM) = T_pM$. Then $(d\exp_p)_0 = id|_{T_pM} : T_pM \to T_pM$.

Pf: LHS $(X_p) = \frac{d}{dt}\Big|_{t=0} \exp_p(tX_p)$

$= \frac{d}{dt}\Big|_{t=0} \gamma(t; p, X_p) = X_p$.

Rmk: $(d\exp)X_p$ isn't no longer id. But we have it's isometry under $g$.

Lemma. (Gauss)

$\langle (d\exp_p)_{X_p} X_p, (d\exp_p) Y_p \rangle_{\exp_p(X_p)} = \langle X_p, Y_p \rangle_p$.

for $X \in \Sigma$. $\forall Y_p \in T_pM$

Cor. $\exists$ nbd $U$ of $0 \in T_pM$ and nbd $V$ of $p \in M$. s.t. $\exp_p : U \to V$ is a diffeomorphism.

Pf: By inverse func. Thm

Rmk: i) $\exp_p$ isn't global diffeo. in general. (e.g. on $S^n$. $\exp_p$ is a diffeo. on $B_r(0) \subset T_pM$. for $\forall\, r < 2$. but fails on $r = 2$)

ii) $\exp_p^{-1} : V \to U$ gives us a coordinate on $M$. around $p$. after identify $T_pM \cong \mathbb{R}^m$.

Def: In Rmk ii) above: Fix o.n.b. $\{e_i\}_1^m$ of $T_pM$. give local coordinate to $U$. Set $\{\tilde{x}_i\}_1^m := \{\exp_p(e_i)\}_1^m$ corrspd. coordinate to $V \subset M$. We call $\{V, \tilde{x}_1 \cdots \tilde{x}_m\}$ normal frame at $p$.

prop. We have $\partial_i|_p = e_i$ under the normal frame. ($\partial_i$ is coord. frame to $\{\tilde{x}_i\}$).

Pf: $\partial_i|_p = D_0 \tilde{x}_i|_p = D_0 \exp_p(e_i) = e_i$.

Cor. $g_{ij}(p) = \delta_{ij}$. $\forall i, j$. under $[\tilde{x}_i]$.

Prop. i) $\Gamma_{ij}^k(p) = 0$. $\forall i, j, k$ ii) $\partial_k g_{ij}(p) = 0$. $\forall i, j, k$.

Pf: i) $\gamma(t) := \exp_p(tX_p)$ geodesic. from $[0,1]$.

for $X_p = x_i \partial_i \overset{\text{Def}}{\underset{\text{of exp.}}{\Longrightarrow}} \gamma : x(t) = (tx_1, \dots tx_m)$

So put in geodesic equation:

$0 = x^k(t)'' + x^i(t)' x^j(t)' \Gamma_{ij}^k(\gamma(t))$.

$= x_i x_j \Gamma_{ij}^k(\gamma(t)) \overset{\text{Set } t=0}{\Longrightarrow} x_i x_j \Gamma_{ij}^k(p) = 0$

for $\forall i, j, k$. $\Rightarrow \Gamma_{ij}^k(p) = 0$. $\forall i, j, k$.

ii) By metric compatible. with i).

We can calculate $\nabla_{\partial_k} \partial_i$. $\nabla_{\partial_k} \partial_j$

explicitly.

Rmk: Taylor expansion of $g_{ij}(x)$ has zero

one-order term at $x = p$ from above.

Actually, under the normal coord.:

$g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{ikej}(p) x_k x_e + O(|x|^3)$

Cor. i) $\nabla_{\partial_i} \partial_j(p) = 0$. (by $\Gamma_{ij}^k(p) = 0$).

ii) $(0, \dots, t\lambda^i, 0 \dots 0)$ is geodesic under $[\tilde{x}_i]$

Def: $B_\varepsilon(p) := \exp_p(\tilde{B}_\varepsilon(0))$ is called geodesic ball of radius $\varepsilon$ center at $p \in M$. And

$S_\varepsilon(p) := \exp_p(\partial \tilde{B}_\varepsilon(0))$. geo-sphere.

Rmk: If $r$ is small enough. Since $\exp_p$ will be a diffeo. Then $B_r(p)$ is a true topological ball.

prop. For $p \in M$. $r$ is sufficiently small.
$\Rightarrow \forall q \in B_r(p)$, $\exists$ unique geodesic connecting $p, q$ with length $< r$.

Pf: Note ~~that~~ the geodesic starting at $p$ with length $< r$ ( $\int_0^1 \|\underbrace{\gamma_t'}_{=\text{const.}}\| = \int_0^1 r < r$ ) lies in $\operatorname{Im} \exp_p$. So it's unique.
$q = \gamma(t; p, X_p)$. So it also exists.

Rmk: i) $B_r(p) \subseteq \hat{B}_r(p)$. $\hat{d}$ is induced metric by Riemann metric $g$

ii) $p, q$ may be connected by other geodesic with longer length (e.g. torus)

iii) It also works for any two pts in strongly geodesically convex nbd.