Ricci Curvature

(1) Definitions:

Def: Ricci tensor $Ric(X,Y) := tr(Z \mapsto R(Z,X)Y)$ is a sym bilinear form.

Rmk: i) $tr(Z \mapsto R(X,Y)Z) = 0$ from skew-sym. $\Rightarrow$ make no sense

ii) $Ric(X,Y) = \sum_i^m \langle R(e_i,X)Y, e_i\rangle$

geometric meaning

$= \sum_1^m S(X \wedge e_i, Y \wedge e_i)$. $X, Y \in T_pM$.

For X unit. (e_k) is o.n.b. of T_pM

$e_m = X$. $\Rightarrow Ric_{ij} = \sum_k R_{ikj}^k$ and

$Ric(X,X) = \sum_1^{m-1} K(\pi_{im})$. where

2-plane $\pi_{im} = span\{e_i, X\}$.

So: $Ric(X,X) = (m-1) \cdot$

$\underset{\pi \ni X}{ave}\, K(\pi)$. over all 2-plane π

in T_pM containing X.

iii) For $m=3$. Ricci curva. Ric_p at p determine all K at p.

It can be interpreted in:
$\Lambda_2 T_pM \simeq T_pM$. We can view
k as quadratic form S on T_pM.

Def: (M,g) has const. Ricci curv. λ at p
if $Ric(X,X) = \lambda_p$ indept of unit vector
$X \in T_pM$. (i.e. $Ric(X_p, Y_p) = \lambda_p g(X_p, Y_p)$).

Thm. (Schur)
(M^n, g) is connected mfd of dim ≥ 3
having const. Ricci curv. at $\forall p \in M$.
Then M has globally const. Ricci curv.
i.e. $\exists \lambda > 0$, $Ric = \lambda g$. $\forall p$ $\forall X, Y \in T_pM$.
Rmk: We call it by Einstein mfd.
Pf: From 2^{nd} Bianchi id.

Thm. (Myers')
(M^n, g) is complete connected mfd with
$Ric \geq {}^{n-1}/_{r^2} \Rightarrow M$ is cpt with diam
$\leq 2r$. "=" holds iff $M = \partial B_r$.

Pf: WLOG. Set $r=1$. By Hopf-Rinow Thm: $\exists$ minimizing unit-speed geodesic γ $: [0,L] \to M$. from p to q.

And $L = d(p,q)$. Next, prove $L \le 2$

Set $\{E_i\}_1^m$ is o.n.b of T_pM. St. $E_m = \gamma'(0)$. and extend each E_i parallel along γ. $\underset{uni}{\Rightarrow}$ $E_m(s) = \gamma'(s)$

Set variation v.f. $V_k(s) = E_k \sin\frac{\pi s}{L}$.

$\Rightarrow \delta_{V_k,V_k} E(\gamma) = \int_0^L \left(\frac{\pi^2}{L^2} - K(\Pi_{km})\right)$ $\sin^2\frac{\pi s}{L}\,ds$ where $\Pi_{km} = \mathrm{span}\{E_k, E_m\}$

since γ is minimizing. So: $\delta_{V_k,V_k} E(\gamma) \ge 0$. $\forall k \le m$.

Sum on $k = 1, \cdots m-1$. we have:

$\int_0^L \left(\frac{(m-1)\pi}{L^2} - Ric(E_m, E_m)\right) \sin^2\frac{\pi s}{L}\,ds \ge 0$.

Complete mfd M with bdd diam is cpt. (Since $M = \exp_p(\bar{B}_d(0))$)

Def: M_k^m is m-dim model space of const. sectional curvature k.

Rmk: i) i.e. M^m_k is $\mathbb{R}^m$, $\mathbb{H}^m$, S^m in properly scaled.

ii) M^m_k has const. Ricci curvature $Ric \equiv (m-1)k$.

Thm. (Bishop - Gromov inequi.)

M^m is complete with $Ric \geq (m-1)k$. $\Rightarrow$ $\forall p \in M$. $r \mapsto \mathrm{Vol}(B_r(p)) / V^m_k(r)$ $\downarrow$ and tend to 1 for $r \downarrow 0$

where $V^m_k(r)$ is volume of geodesic ball of radius r in M^m_k.

Rmk: $Ric \uparrow$, Vol. of r-ball $\downarrow$.

Def. A geodesic line in M is geodesic $\gamma : \mathbb{R}^1 \to M$. st. every subarc is length minimizing. (e.g. $\forall$ geodesic in $\mathbb{R}^m$, $\mathbb{H}^m$)

Thm. (Cheeger - Gromoll Splitting Thm.)

If M is complete mfd with $Ric \geq 0$ and contains a geodesic line. Then M is isometric to $N \times \mathbb{R}^1$.

Thm. (Upper bdd for Ricci.)

Any M^m with $m \geq 3$ admits a metric of negative $Ric < 0$.

Rmk: By Gauss-Bonnet Thm. for $m = 2$, the sphere has no metric of negative Ricci curvature.

Def: Ricci flow $\partial g_{ij}/\partial t = -2 Ric_{ij}$ is a non-linear heat flow for (M, g). Which tries to smooth out Ric. curv.

Rmk: It can be used to prove the geometric conjecture for 3-mfd (Decomposition of 3-mfd.)

(2) Scalar Curvature:

Def: Scalar curvature $S \in C^\infty(M)$ of (M, g) is $S \stackrel{\Delta}{=} tr_g Ric := \sum_i Ric_i^i = \sum_{ij} g_{ij} Ric_{ij}$ where $Ric_i^j = \sum_k g_{jk} R_{ik}$ trace of Ricci curvature w.r.t. g.

Rmk: If $\{E_i\}$ is choosen to be local ortho. frame $\Rightarrow g_{ij} = \delta_{ij}$.

So: $S = \sum_i Ric(E_i, E_i)$.

$= m \cdot \text{avg } Ric(E, E)$. $E \in T_pM$ unit.

$= m(m-1) \operatorname{avg}_{i,j} K(\Pi_{ij})$

Thm. (Yamabe problem)

Given cpt mfd (M^m, g_0). $m \geq 3$. $\exists$ conformally equi. $g = e^{2\varepsilon} g_0$. admits a const. scalar curvature.

Rmk: Those is the metric with const. scalar curvature are critical pt for total scalar curva. under volum-preserving conformal change.

Thm. $\forall$ mfd M^m with $m \geq 3$ admits a metric of const negative scalar curv.

Rmk: It fails when M is sphere or torus.

(3) Computation on Ricci:

under normal coordinate $(U; x_1 \cdots x_m)$:

The volume factor is

$$\sqrt{\det g_{ij}(x)} = 1 - \frac{1}{6}\sum_{k,\ell} \mathrm{Ric}_{k\ell}\, x^k x^\ell + O(|x|^3).$$

So along a geodesic $\gamma(t) = \exp_p(tV)$ with $|V| = 1$ we have

$$\sqrt{\det g_{ij}(\gamma(t))} = 1 - \frac{\mathrm{Ric}(V,V)}{6}t^2 + O(t^3).$$

If α_m denotes the volume of the unit ball in $\mathbb{R}^m$, then the volume of a geodesic ball of radius r is given by

$$\mathrm{vol}(B_r(p)) = \alpha_m r^m \underline{\left(1 - \frac{S(p)}{6(m+2)}r^2 + O(r^3)\right)}.^{(*)}$$

Similarly, for the $(m-1)$-dimensional area of the sphere,

$$\mathrm{area}(S_r(p)) = m\alpha_m r^{m-1}\underline{\left(1 - \frac{S(p)}{6m}r^2 + O(r^3)\right)}.^{(*')}$$

In particular for a surface M^2, we have $\alpha_2 = \pi$ and $S = 2K$, so the circumference of a geodesic circle of radius r is

$$2\pi r - \frac{\pi}{3}K(p)r^3 + O(r^4),$$

giving another intrinsic interpretation of the Gauss curvature $K(p)$.

Rmk: $(*)$. $(*')$ acts as a distortion coeff in formula of geodesic ball/sphere.