Tangent Vec. & Submfd

(1) Smooth Maps and Tangent:

Def: i) M, N are m, n-dim manifolds with diff. structure $\mathcal{A}_M, \mathcal{A}_N$ of C^k.

For conti. map $f: M \to N$. is said C^r-class. for $r \leq k$. if for any $(U, x) \in \mathcal{A}_M$, $(V, y) \in \mathcal{A}_N$.

$$\mathbb{R}^m \overset{\text{open}}{\supseteq} x(U \cap f^{-1}(V)) \xrightarrow{y \circ f \circ x^{-1}} y(V) \overset{\text{open}}{\subseteq} \mathbb{R}^n.$$

is of class C^r.

Rmk: f is conti. is for the differentiality can be def on open set

ii) $f \in C^r(M; N)$ is C^r-diffeomorphism if f is bijective, $f, f^{-1} \in C^r$.

And if $f(\lambda v + w) = \lambda f(v) + f(w)$. i.e.
reserving linear operation. then:
we call it v.s. diffeomorphism.

Def: $p \in M$. M is smooth manifold.

i) A path through p in M is a smooth map $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$. st. $\gamma(0) = p \in M$

ii) Two paths $\gamma, \tilde{\gamma}$ defined in i) are tangent if $\exists$ some chart (U, x). st. $\forall\, p \in U$, we have: at $t = 0$.
$d/dt\,(x \circ \gamma) = d/dt\,(x \circ \tilde{\gamma})$

Rmk: It's indept of the choice of chart:

$$(\eta \circ \alpha)'(0) = D(\eta \circ x^{-1})(x(p))\,(x \circ \alpha)'(0)$$
$$= \sim (x \circ \beta)'(0)$$
$$= (\eta \circ \beta)'(0).$$

iii) Tangent vector to M at p is the equi. class $[\gamma]$. of path through p in M tangent with each other

iv) Tangent space to M at p is:

$T_pM := \{ [\gamma] \mid \gamma : \text{path in iii)} \}$.

<u>prop.</u> T_pM has a unique v.s. structure s.t. $\forall$ smooth n-dim chart (U, x). satisfy $\forall p \in U$. $d_p x : T_pM \to \mathbb{R}^n$ is a
$[\gamma] \mapsto (x \circ \gamma)'(0)$
v.s. isomorphism.

<u>Pf</u>: 1) Bijection: Let $\gamma(t) = x^{-1}(tv + p)$ for $\forall v \in \mathbb{R}^n$.

2) Note $d_p y \circ (d_p x)^{-1} = D(y \circ x^{-1})(x(p))$ is v.s. isomorphism $: \mathbb{R}^n \to \mathbb{R}^n$.

<u>Rmk</u>: $\dim M = \dim T_pM$. if dim exists.

Eg $M \subseteq_{opn} V$. n-dim vector space. $\Rightarrow$ We can just define: $T_pM \xrightarrow{\sim} V : [\gamma] \mapsto \dot{\gamma}(0)$.

Rmk: We will use this canonical map. to identify Tangent space on open subset of n-dim. v.s. V as V itself.

Def: Cotangent space is $T_p^*M := L(T_pM, \mathbb{R})$.

Def: i) Tangent bundle TM of M is:

$$TM := \bigcup_{p \in M} T_pM.$$

Rmk: $\forall p$. T_pM has a zero vector $[0]$. called zero section. (i.e. $\gamma \equiv p$).

ii) Tangent projection: $\pi : TM \to M$. defined by $\pi^{-1}(p) = T_pM$.

iii) Natural inclusion $i: \begin{array}{l} M \to TM. \\ p \mapsto [0] \in T_pM. \end{array}$

We also have: $\tilde{j}: T_pM \hookrightarrow TM.$
$v \mapsto (q, v).$

Lemma. n-dim chart (U, x) determines a $2n$-dim chart (TU, Tx) on TM. st $TU \triangleq \bigcup_{p \in U} T_pM$. $U \overset{\text{open}}{\subseteq} M$. and that.

$$T_x: \begin{array}{ccc} TU & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \\ [\gamma] \in T_pM & \longmapsto & (x(p), d_px([\gamma])) \end{array}$$

For (TV, Ty) another chart. corspd (V, y). We have $Ty \circ (Tx)^{-1}(q, v) = (y \circ x^{-1}(q), D(y \circ x^{-1})(q) \cdot v)$

Rmk: T_x record the information of the point and its tangent.

Pf: $T_x(TU) = x(U) \times \mathbb{R}^n \underset{\text{open}}{\subseteq} \mathbb{R}^{2n}$.

and it's easy to check it's smooth.

Cor. TM can endow naturally the smooth structure of M and be $2n$-dim manifold. s.t. π and $i, \tilde{i}$ are smooth maps.

Pf: By Lemma above, we only need to check metrizable and separable.

The former is from setting of Riemann metric

The latter: $D \underset{\text{dense}}{\overset{\text{countable}}{\subset}} M$.

since $T_pM \cong \mathbb{R}^n$ is separa.

$\Rightarrow \bigcup_{p \in D} T_pM$ is separable.

Rmk: If M is C^k-manifold. Then: TM will be C^{k-1}. since it involves 1-order derivative!

Def: Cotangent bundle $T^*M = \bigcup_{p \in M} T_p^*M$.

With tangent bundle, we can define the derivative of $f \in C^k(M, N)$:

Def: For $f \in C^\infty(M, N)$, the tangent map of f is $Tf : TM \to TN$.
$[\gamma] \mapsto [f \circ \gamma]$.

The derivative at p is:

$T_p f : T_p M \to T_{f(p)} N$. $[\gamma] \mapsto [f \circ \gamma]$.

Lemma. $T_p f$ is linear map. Which's indpt of choice of $\gamma \in [\gamma]$. Besides. $f \in C^\infty$ $\Rightarrow Tf \in C^\infty$.

Pf: Choose chart (TU, Tx). (TV, Ty) rised from (U, x), (V, y).

Note $Ty \circ Tf \circ Tx^{-1}(x(p), (x \circ \gamma)'(0))$

$$= (\underbrace{y \circ f \circ x^{-1}(x(p))}_{= p}, D(y \circ f \circ x^{-1})(x(p)) \cdot \underbrace{(x \circ \gamma)'(0)}_{= v})$$

Prop. (Chain Rule)

$f \in C^\infty(M, N)$. $g \in C^\infty(N, G)$. Then:

$T(f \circ g) = Tf \circ Tg : TM \to TG$.

Cor. $(Tf)^{-1} = T(f^{-1})$.

Pf: $id = T(f \circ f^{-1}) = Tf \circ T(f^{-1})$.

(2) Submanifolds:

① 1) Consider $U \overset{open}{\subset} \mathbb{R}^n$. M is manifold. And $f \in C^\infty(U; M)$. Define partial derivative

$\partial_j f(x_0) = \frac{\partial f}{\partial x^j}(x_0) := [\gamma_j] \in T_{f(x_0)}M$. s.t.

$\gamma_j = f(x_0 + te_j) = f(x_0^1 \cdots x_0^{j-1}, x_0^j + t, \cdots x_0^n)$.

Rmk: $(\partial_j f(x_0))_1^n$ has same info. as the tangent map $T_{x_0}f : T_{x_0}U \to T_{f(x_0)}M$.

2) Consider V is n-dim v.s. M is manifold. $f \in C^\infty(M, V)$. Note $T_{f(p)}V = V$. $\Rightarrow$

Set $df : TM \to V$. $\quad d_pf : T_pM \to V$

$[\gamma] \mapsto (f \circ \gamma)'(0) \qquad [\gamma] \mapsto (f \circ \gamma)'(0)$

Rmk: If $V = \mathbb{R}^1$. Then: $d_pf \in T_p^*M$.

More generally, for $d_px : T_pM \to \mathbb{R}^n$.

We can see X through $(X^1 \cdots X^n)$.

$\Rightarrow d_p X = (d_p X^1, \cdots d_p X^n). \in \mathbb{R}^n$.

It's easy to see $d_p X$ contains the same info as $T_{x(p)} V \cong U$.

② Inverse Func Thm:

THEOREM (inverse function theorem). *Suppose $\mathcal{U} \subset \mathbb{R}^n$ is open, $f : \mathcal{U} \to \mathbb{R}^n$ is a map of class C^k for some $k \in \mathbb{N} \cup \{\infty\}$, and $x_0 \in \mathcal{U}$ is a point at which the derivative $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Then there exist open neighborhoods $x_0 \in \Omega \subset \mathcal{U}$ and $f(x_0) \in \Omega' \subset \mathbb{R}^n$ such that f maps Ω bijectively onto Ω' and the inverse $(f|_\Omega)^{-1} : \Omega' \to \Omega$ is also of class C^k.* □ full rank

Lemma': $\mathcal{U} \underset{\text{open}}{\subset} M$. smooth n-manifold. $\varphi \in C^\infty(\mathcal{U} : M)$. $x_0 \in \mathcal{U}$. s.t. $(\partial_j \varphi(x_0))_1^n$ form a basis of $T_{\varphi(x_0)} M$. $\Rightarrow \exists$ nbd Ω of $x_0 \subset \mathcal{U}$. and O of $\varphi(x_0) \subset M$, s.t. $\varphi : \Omega \xrightarrow{\sim} O$. $(O. (\varphi|_\Omega)^{-1})$ is a chart on M.

Pf: Choose (V, y). chart on M.

$\Rightarrow d_p y (\partial_j \varphi(x_0)) = D_j (y \circ \varphi)(x_0)$

form a basis in $\mathbb{R}^n$.

So: $D(\eta \circ \varphi)$ is isomorphism.

$\Rightarrow$ Apply IFT. to find nbds.

<u>Lemma</u>[2]: M is smooth n-manifold. $\exists\, U$. open. If

$x^1 \cdots x^n : U \to \mathbb{R}^1$. smooth. $p \in U$. st $(d_p x^i)_1^n$

form a basis of $T_p^* M$. Then: $\exists$ nbd

O of $p \subset U$. st. (O, x) is smooth

chart on M.

<u>pf</u>: Note $\exists\, (X_k)_1^n \subset T_p M$. st.

$d_p x^j (X_k) = \delta_{kj}$. (send basis to basis)

Next, set $d_p x = (d_p x^1 \cdots d_p x^n)$

$: T_p M \xrightarrow{\sim} T_{x(p)} \mathbb{R}^n = \mathbb{R}^n$. $\Rightarrow$ isomorp.

Choose another chart (V, η).

$\Rightarrow D(x \circ \eta^{-1})(\eta(p)) = d_p x \circ (d_p \eta)^{-1}$ is

still isomorphism. $\Rightarrow$ Apply IFT.

(8) **Def:** i) A chart (U, x) on n-manifold M is l-dim slice chart for $L \subset M$ if $L \cap U \subset x^{-1}(\mathbb{R}^l \times \{0\})$.

ii) $L \subset M$ n-manifold is l-dimen. submfd if M has a collection of slice charts $\{(U_\alpha, x_\alpha)\}$. s.t. $L = \bigcup U_\alpha$.

Prop. If L is l-dim C^k-submanifold of n-dim C^k-manifold M. Then L inherits naturally the structure from M. s.t. $L \overset{i}{\hookrightarrow} M$ is smooth

Besides, T_pL is l-dim LS of T_pM.

Pf: i) For slice chart (U, x) of M. Set $x_L = \pi_l \circ x|_{U \cap L}$ from $L \cap U$ to $\mathbb{R}^l$.

Then $\{(U \cap L, x_L)\}$ form a C^k-chart of L.

With L is also metriz. and separ. since it's subset of M $\Rightarrow$ L is ℓ-manifold.

Note $x \circ i \circ x_L^{-1} = (x^1 \ldots x^\ell) \mapsto (x^1 \ldots x^\ell, 0 \ldots 0)$.

is smooth as well.

ii) $T_p i : T_p L \to T_p M$ is canonical inclusion.

__Def__: $f \in C^\infty(M, N)$ is immersion / submersion at p if $T_p f$ is injective / surjective.

__Rmk__: i) Denote immersion by $f : M \looparrowright N$. Which means an immersion may not be injective (e.g. Klein bottle)

ii) Set of immersions / submersions is open.

iii) It turns out up to the choice of coordinates at $p \in M$, $f(p) \in N$. All

Immersions / submersions look same.

Thm. (Rank Thm)

$f \in C^\infty(M, N)$. M, N: smooth m, n-mani. $p \in M$, $q = f(p) \in N$. If f is immersion or submersion. Then $\exists$ smooth chart (U, x) on M, (V, y) on N s.t. $x(p) = 0$, $y(q) = 0$. and

$$y \circ f \circ x^{-1} : (x^1 \dots x^m) \mapsto \begin{cases} (x^1 \dots x^n), & m \geq n \text{ (submersion)} \\ (x^1 \dots x^m, 0 \dots 0) & m \leq n \text{ (immersion)} \end{cases}$$

Pf: 1) If $T_p f$ is injective. $l = n - m$

WLOG. $\exists (U, x)$ on M. $x(p) = 0$. Set $F = f \circ x^{-1} : \Omega \to N$. $\Omega = x(U) \subseteq \mathbb{R}^m$

$\Rightarrow T_0 F = T_p f \circ (d_p x)^{-1}$ is injective

i.e. $(\partial_i F(0))_1^m$ is l.i. in N.

choose chart $(\tilde{V}, \tilde{y})$ on N. $\tilde{V} \ni q$

s.t. $\tilde{F} = \tilde{y} \circ F : \Omega \to \tilde{N} \subseteq \mathbb{R}^n$.

We can replace F by $\tilde{F}$ in argument above

Extend the l.i. set $(\partial_i \tilde{F}(0))_1^m$

to $(\partial_i \tilde{F}(0))_1^m \cup (Y_k)_{m+1}^n$ on $T_{\tilde{\eta}(q)}\tilde{N}$.

Set $G(x_1 \cdots x^n) = \tilde{F}(x^1 \cdots x^m) + \sum_{m+1}^{n} x^j Y_j$

$: \Omega \times (-\varepsilon, \varepsilon)^l \to \tilde{N}$. ε suff. small

Apply the Lem in ① on $\tilde{\eta}^{-1} \circ G$.

We can set $\eta = (\tilde{\eta}^{-1} \circ G)^{-1}$

2) If $T_p f$ is surjective. $l = m - n$

Choose chart (V, η) on N. $\eta(q) = 0$.

Set $\hat{x} = (x^1 \cdots x^n) = f^{-1}(V) \to \mathbb{R}^n$.

where $x^i = \eta^i \circ f$.

$\Rightarrow d_p \hat{x} = d_q \eta \circ T_p f$ is surjective

So: $d_p x^1 \cdots d_p x^n$ is l.i. in $T_p^* M$.

extend to $(d_p x^k)_1^n \cup (\Lambda_k)_{n+1}^m$ (basis)

Set $x^i(p) = 0$, $d_p x^i = \Lambda^i$. $n+1 \le i \le m$.

follows from see in local chart.

Set $x = (x^1 \cdots x^m)$.

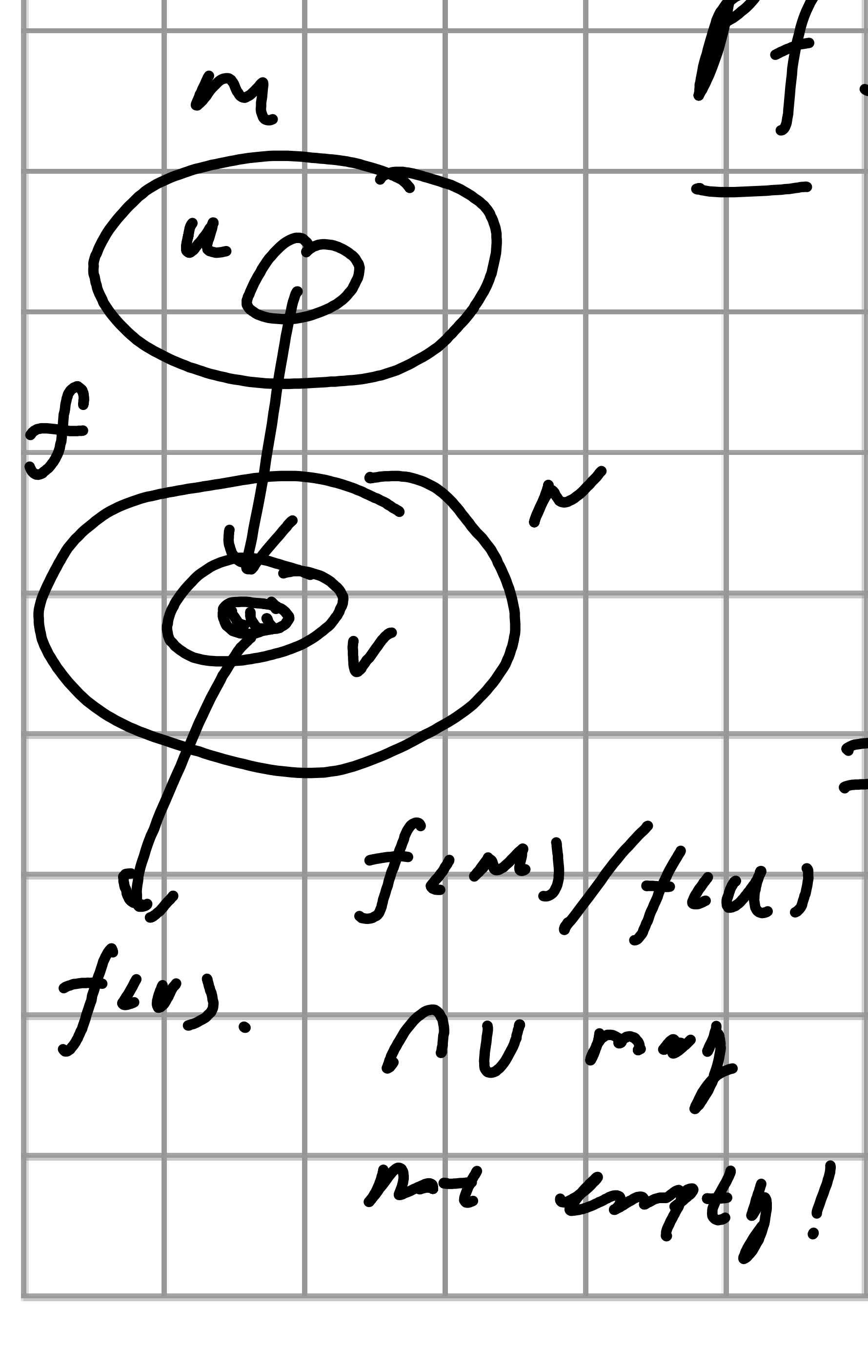
④ Level sets:

Def: $f \in C^\infty(M, N)$ is embedding if it's a injective immersion s.t. f^{-1} is also conti.

Lemma Every immersion is locally injective

Pf: By the rank Thm: compose with chart (y, V). (x, U) locally we have a coordinate mapping.

Thm. If $f: M \to N$ is embedding. Then $f(M)$ is smooth submanifold of N.



Pf: For $q \in f(M)$. $f(p) = q$.

By rank Thm in ③. We have (U, x) in M. (V, y) on N.

$\Rightarrow f(U)$ is open in $f(M)$. So:

$\exists W$ open in N. $f(U) = f(M) \cap W$

$$\Rightarrow \underbrace{\tilde{W}}_{=W\cap V} \cap f(m_0) \underset{\text{open}}{\subset} f(W) = \underbrace{y}_{\cap V}^{-1}(y \circ f \circ x^{-1}(x(\tilde{m}_0)))$$

i.e. $(V \cap W, y)$ is slice chart.

<u>Cor.</u> $L \subset M$. smooth manifold is its smooth submanifold $\Leftrightarrow$ it admits smooth structure. s.t. $L \hookrightarrow M$ is smooth embedding

<u>Pf:</u> Combined with prop. in ⑧

<u>e.g.</u> $f: M = \mathbb{R} \sqcup (0,2) \to \mathbb{R}^2$ defined by

$f(t) = (t, 0)$. $t \in \mathbb{R}'$
$f(\theta) = (\cos\theta, \sin\theta)$. $\theta \in (0,2)$ $\Rightarrow f$ is a

injective immersion. But f^{-1} isn't

conti at $(1,0)$. $\lim_{z \to (1,0)} f^{-1}(z) = \underset{\in \mathbb{R}'}{1}$ or $\underset{\in [0,2]}{0}$.

<u>Rmk:</u> But if X is cpt. Y is Hausdorff

$\Rightarrow$ conti. bijection $f: X \to Y$ is homeo.

<u>Pf:</u> Check f is close map.

It's obvious since $f(A) \overset{cpt}{\subset} Y$ if A close

Cr. Any $f: X \to Y$. conti. injection is also embedding. for X, Y. t.s.

Cr. $f: M^m \to N^n$. injective immersion. if M is cpt mfd. then: f is smooth embedding

Def: $f \in C^\infty(M, N)$. $p \in M$ is regular pt if f is submersion at p. and it's critical pt otherwise. And we call $f(p)$ is regular / critical value.

Thm. (IFT).

$f \in C^\infty(M, N)$ with regular value $q \in N$. $\Rightarrow$ $L = f^{-1}(q)$ is smooth sub-mfd with $\dim L = \dim M - \dim N$. And $T_pL = \ker T_pf \subset T_pM$.

$f^{-1}(q)$ is called regular level set

Pf: (i) Apply the rank Thm:

$\exists (U,x), (V,y)$ on M, N. $x(p) = y(q) = 0$.

$\Rightarrow f^{-1}(q) = x^{-1}(x\circ f^{-1}\circ y^{-1})(\underbrace{y(q)}_{=0})$

$= x^{-1}(\mathbb{R}^m \times \{0\})$

i.e. (U,x) is slice chart.

ii) Note $T_pf: T_pM \to T_{f(p)}N$.

$\forall \gamma \in f^{-1}(q)$, $\gamma \equiv$ const. $\Rightarrow \gamma \in [0]$

So $T_pL \subset \ker T_pf$. And $\dim T_pL$

$= \dim L = \dim M - \dim N$

$= \dim T_pM - \dim T_pN = \dim \ker T_pf$.

Rmk:

Submanifolds can be described by: $(m \le n)$

i) zero level set of submersion $N^n \to M^m$

ii) Image of embedding: $M^m \to N^n$.